

Pisot-Vijayaraghavan numbers

A Pisot-Vijayaraghavan number, also called simply a Pisot number or a PV number, is a real algebraic integer greater than 1 such that all its Galois conjugates are less than 1 in absolute value. These numbers were discovered by Axel Thue in 1912 and rediscovered by G. H. Hardy in 1919 within the context of diophantine approximation.

They became widely known after the publication of Charles Pisot's dissertation in 1938. The same condition also occurs in the uniqueness problem for Fourier series. Tirukkannapuram Vijayaraghavan and Raphael Salem continued their study in the 1940s. Salem numbers are a closely related set of numbers.

A characteristic property of PV numbers is that their powers approach integers at an exponential rate. Pisot proved a remarkable converse: if $\alpha > 1$ is a real number such that the sequence $\epsilon_n = \inf_{k \in \mathbb{Z}} |\alpha^n - k|$ measuring the distance from its consecutive powers to the nearest integer is square-summable, then α is a Pisot number (and, in particular, algebraic). Building on this characterization of PV numbers, Salem showed that the set S of all PV numbers is closed. Its minimal element is a cubic irrationality known as the plastic number. Much is known about the accumulation points of S . The smallest of them is the golden ratio.

An algebraic integer of degree n is a root α of an irreducible monic polynomial $P(x)$ of degree n with integer coefficients, its minimal polynomial. The other roots of $P(x)$ are called the conjugates of α . If $\alpha > 1$ but all other roots of $P(x)$ are real or complex numbers of absolute value less than 1, so that they lie strictly inside the circle $|z| = 1$ in the complex plane, then α is called a Pisot number, Pisot-Vijayaraghavan number, or simply PV number. For example, the golden ratio, $\phi \sim 1.618$, is a real quadratic integer that is greater than 1, while the absolute value of its conjugate, $\bar{\phi} \sim -0.618$, is less than 1. Therefore, ϕ is a Pisot number. Its minimal polynomial is $x^2 - x - 1$.

Elementary properties:

(a) Every positive integer except 1 is a PV number. Conversely, every rational PV number is an integer greater than 1.

(b) If α is an irrational PV number whose minimal polynomial ends in k then α is greater than $|k|$. Consequently, all PV numbers that are less than 2 are algebraic units.

(c) If α is a PV number then so are its powers α^k , for all natural exponents.

(d) Every real algebraic number field \mathcal{K} of degree n contains a PV number of degree n . This number is a field generator. The set of all PV numbers of degree n in \mathcal{K} is closed under multiplication.

(e) Given an upper bound M and degree n , there is only a finite number of PV numbers of degree n that are less than M .

Diophantine properties:

The main interest in PV numbers is due to the fact that their powers have a very “biased” distribution (mod 1).

If α is a PV number and λ is any algebraic integer in the field of α then the sequence $\|\lambda\alpha^k\|$ where $\|x\|$ denotes the distance from the real number x to the nearest integer, approaches 0 at an exponential rate. In particular, it is a square-summable sequence and its terms converge to 0. Two converse statements are known: they characterize PV numbers among all real numbers and among the algebraic numbers (but under a weaker diophantine assumption).

(1) Let α be a real number greater than 1 and suppose that there exists a non-zero real number λ such that

$$\sum_1^{\infty} \|\lambda\alpha^k\|^2 < \infty.$$

Then α is a Pisot number and λ is an algebraic number in the field of α (Pisot's theorem).

(2) Let α be an algebraic number greater than 1 and suppose that there exists a non-zero real number λ such that $\|\lambda\alpha^k\|$ tends to 0 as k tends to infinity.

Then α is a Pisot number and λ is an algebraic number in the field of α .

A longstanding Pisot-Vijayaraghavan problem asks whether the assumption that α is algebraic can be dropped from the last statement. If the answer is affirmative, Pisot's numbers would be characterized among all real numbers by the simple convergence of $\|\lambda\alpha^k\|$ to 0 for some auxiliary real λ . It is known that there are only countably many numbers α with this property. The problem is to decide whether any of them is transcendental.

Topological properties:

The set of all Pisot numbers is denoted S . Since Pisot numbers are algebraic, the set S is countable. Raphael Salem proved that this set is closed: it contains all its limit points. His proof uses a constructive version of the main diophantine property of Pisot numbers: given a Pisot number α , a real number λ can be chosen so that $0 < \lambda \leq \alpha$ and

$$\sum_{k=1}^{\infty} \|\lambda \alpha^k\|^2 \leq 9.$$

Thus the l^2 norm of the sequence $\|\lambda\alpha^k\|$ can be bounded by a uniform constant independent of α . In the last step of the proof, Pisot's characterization is invoked to conclude that the limit of a sequence of Pisot numbers is itself a Pisot number.

Closedness of S implies that it has a minimal element. Carl Ludwig Siegel proved that it is the positive root of the equation $x^3 - x - 1 = 0$ (plastic constant) and is isolated in S .

He constructed two sequences of Pisot numbers converging to the golden ratio ϕ from below and asked whether ϕ is the smallest limit point of S . This was later proved by Dufresnoy and Pisot, who also determined all elements of S that are less than ϕ ; not all of them belong to Siegel's two sequences. Vijayaraghavan proved that S has infinitely many limit points; in fact, the sequence of derived sets S, S', S'', \dots does not terminate.

On the other hand, the intersection $S^{(\omega)}$ of these sets is empty, meaning that the Cantor-Bendixson rank of S is ω .

A real algebraic integer $\alpha > 1$ is a Salem number if all its conjugate roots have absolute value no greater than 1, and at least one has absolute value exactly 1. Salem numbers are named after Raphael Salem.

The set of Salem numbers, denoted by T , is intimately related with S .

It has been proved that S is contained in the set T' of the limit points of T .

It has been conjectured that the union of S and T is closed.