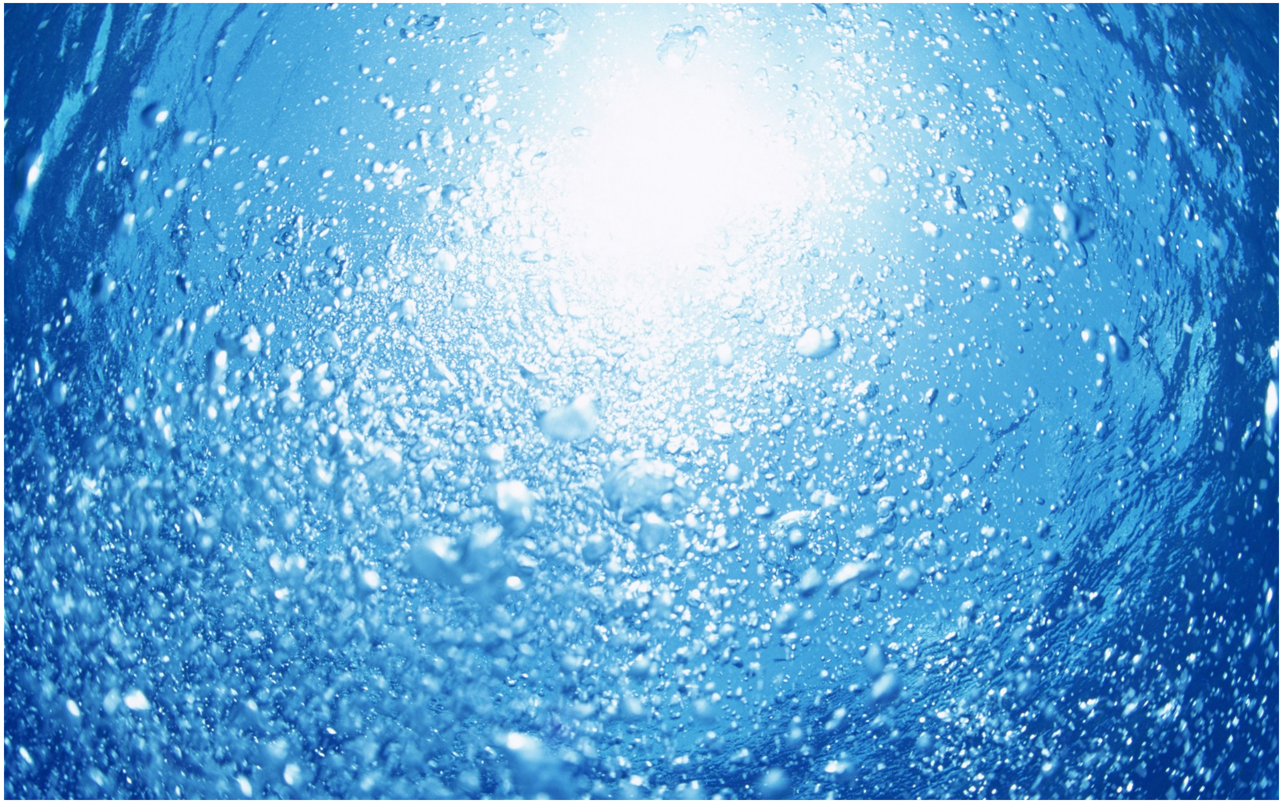
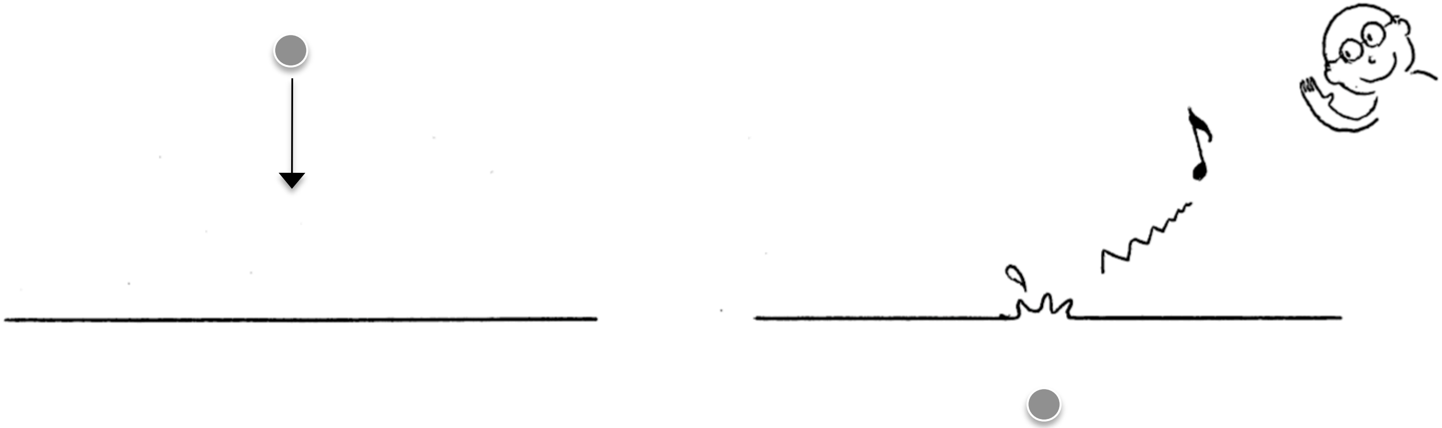
An invitation to simple modeling of complex phenomena



T. Tokieda
Lyon, August 2012

Which ***musical note*** does a projectile make on splashing into water ?



Of the three approaches to modeling,
3) solving the full equation is impossible because the full equation is unknown.

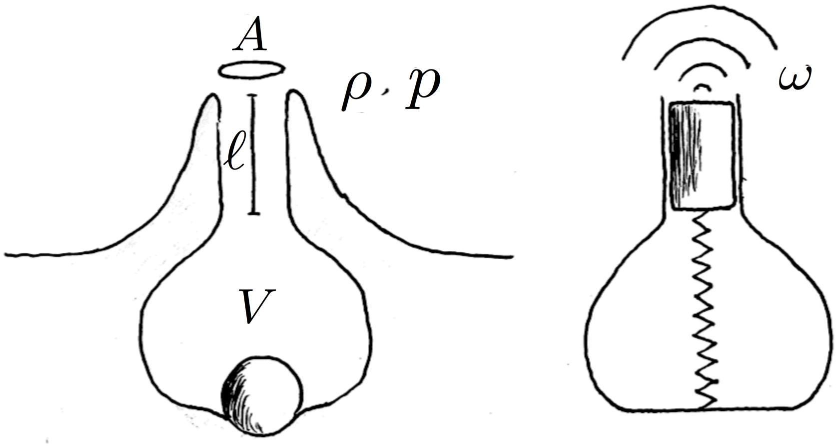
So we must resort to 1) dimensional analysis or
2) back-of-the-envelope estimate.

We will do 2), then check it by 1).

2) *Back-of-the-envelope estimate*

What produces the sound ?

The projectile creates a cavity in water.
Air in the *bulb* acts like a ***spring*** while
air in the *neck* acts like a mass.



Let $\Delta z =$ vertical displacement of neck air

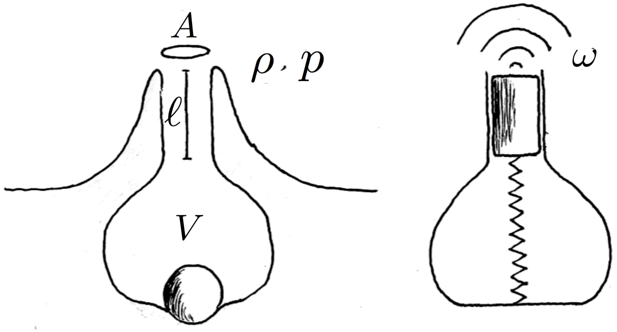
$\Delta p =$ pressure difference outside/inside cavity

Compressibility of air is

By thermodynamics it is also

$$\left.\begin{array}{l} -\dfrac{1}{V}\dfrac{\partial V}{\partial p} \approx -\dfrac{1}{V}\dfrac{A\Delta z}{\Delta p} \\[2ex] \dfrac{1}{\gamma p} \quad \text{where } \gamma \approx 1.4 \end{array}\right\}$$

$$\Longrightarrow \quad \Delta p \approx -\frac{\gamma\, pA}{V}\Delta z$$



$$\Delta p \approx -\frac{\gamma\, p A}{V}\Delta z$$

Therefore the spring equation is

$$\rho\,\ell A \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Delta z = A\Delta p \approx -\frac{\gamma\, p A^2}{V}\Delta z$$

mass × accel force

$\Longrightarrow$ angular frequency $\boxed{\omega \approx c\sqrt{\frac{A}{V\ell}}}$

where $c = \sqrt{\frac{\gamma\, p}{\rho}} \approx 345\,\mathrm{m/sec}\ \ (\mathrm{at}\ \ 22^\circ\mathrm{C})$

is the speed of sound in air.

Our formula

$$\boxed{\omega \approx c\sqrt{\frac{A}{V\ell}}}$$

passes the test of **1)** *dimensional analysis*

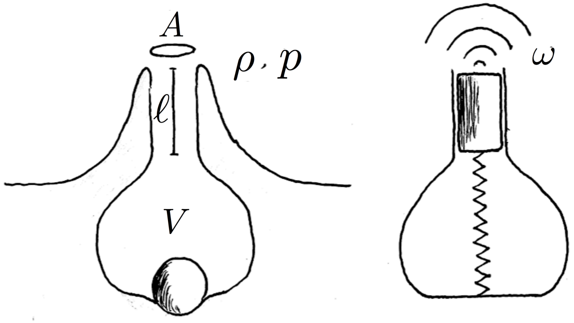
Take a stone of diameter $d \approx 3\times10^{-2}\,\mathrm{m}$.

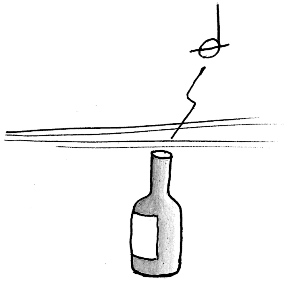
$$\ell \approx d \quad A \approx d^2 \quad V \approx (2d)^3$$

predict

$$\frac{\omega}{2\pi} \approx \frac{345}{2\pi\sqrt{8}\,d} \approx 647\,\mathrm{Hz}$$

or **E$_5$** (middle C is C$_4$).





Blowing across the top of a bottle has the same physics.

$$\ell \approx 8\,\mathrm{cm} \quad A \approx \pi\,\mathrm{cm}^2 \quad V \approx 700\,\mathrm{cm}^3$$

predict

$$\frac{\omega}{2\pi} \approx 130\,\mathrm{Hz}$$ or **C$_3$** .

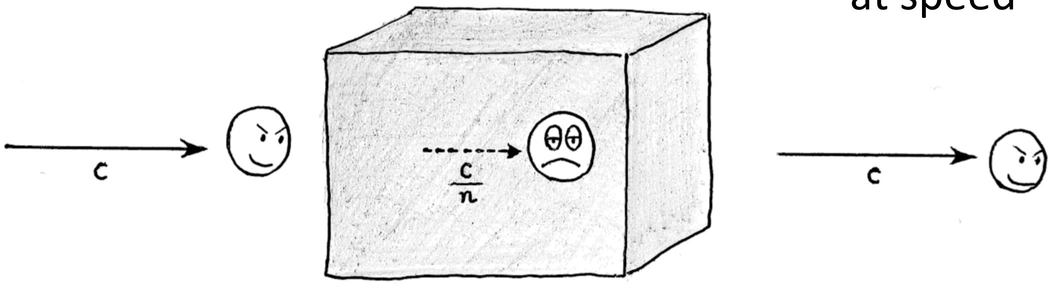
We have been studying oscillators
with forcing (tide, pendulum) and
without (surface wave on water, sound),
but we have not yet considered ***dissipation*** .

We now consider phenomena involving dissipation (light).

Problem of refractive index

When passing through a material medium (air, water, glass, . . .), light *appears* to travel slower,

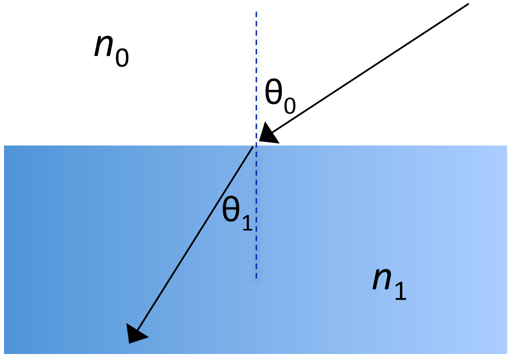
at speed $\frac{c}{n}$ instead of c.



This factor n is the ***refractive index*** .

It underpins Snell's law, law of reflection — all of geometric optics.

$$n_0 \sin \theta_0 = n_1 \sin \theta_1$$



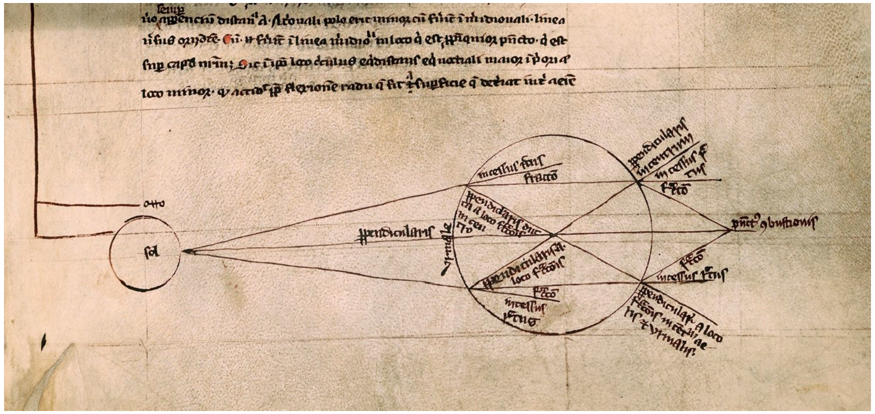
Can we estimate n from atomic data ?

Aside :

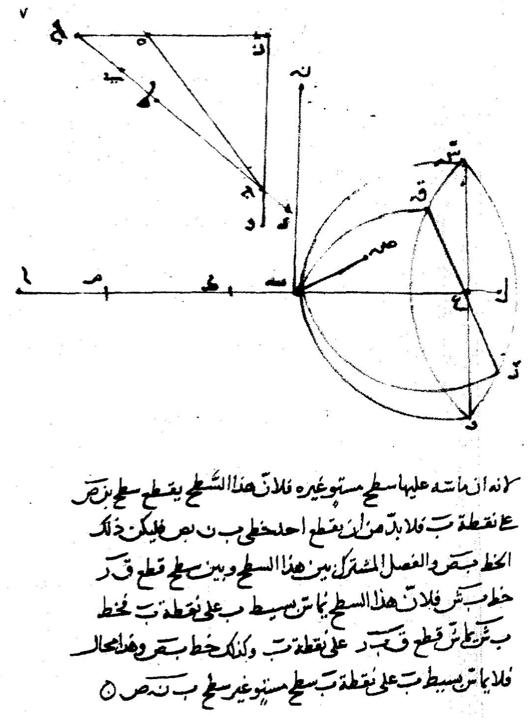
Snell’s law is called loi de Descartes by the French.

Pointless to debate the priority between these 17th-century gentlemen, since the law was already published by

[Roger Bacon 1267]



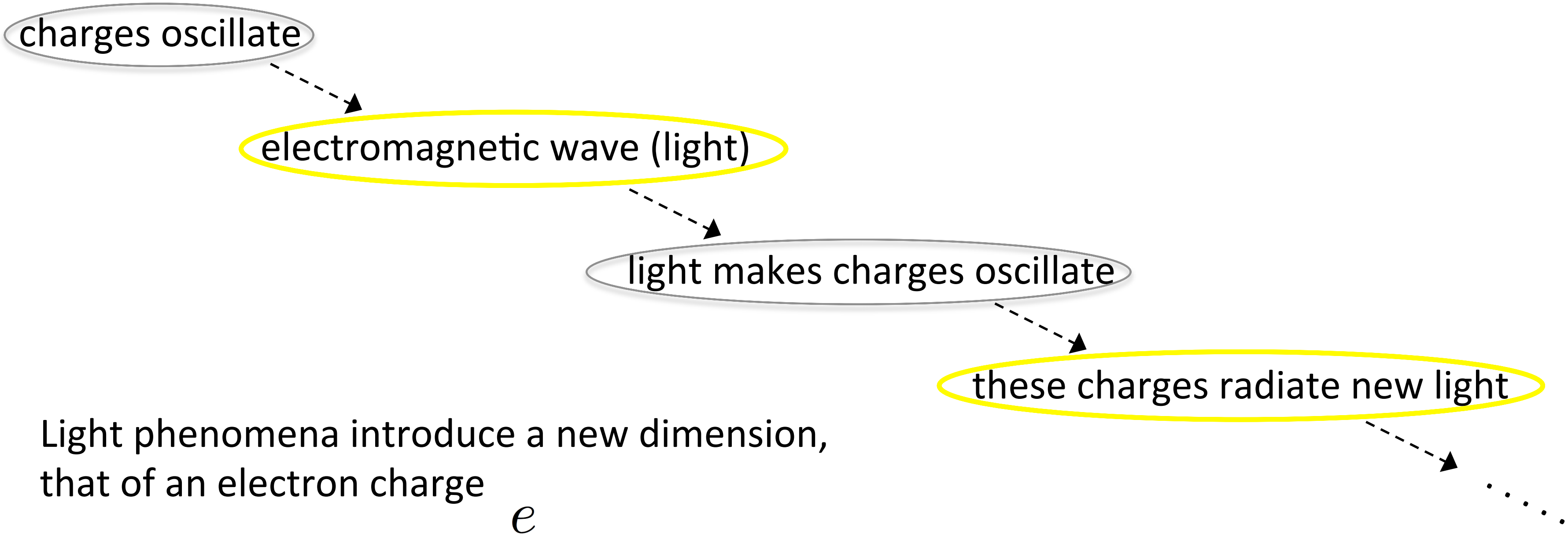
who in turn copied it from



لانه ان ماسته عليها سطح مستو غيره فلان هذا السطح يقطع سطح ب ن ص
على نقطة ب فلا بد من ان يقطع احد خطي ب ن ب ص فليكن ذلك
الخط ب ص والفصل المشترك بين هذا السطح وبين سطح قطع ق ر
خط ب ش فلان هذا السطح يماس بسيط ب على نقطة ب فخط
ب ش يماس قطع ق ب د على نقطة ب وكذلك خط ب ص وهذا محال
فلا يماس بسيط ب على نقطة ب سطح مستو غير سطح ب ن ص ○

[Ibn Sahl 984]

How light propagates :



charges oscillate

electromagnetic wave (light)

light makes charges oscillate

these charges radiate new light

. . . . .

Light phenomena introduce a new dimension,
that of an electron charge

$$e$$

In practical problems, better to work with

$$\frac{e^2}{4\pi\varepsilon_0}$$

$$\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}$$ is a force, $$\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{r}$$ a potential

$$\Longrightarrow \boxed{\left[\frac{e^2}{4\pi\varepsilon_0}\right]=\frac{\mathbf{M}\,\mathbf{L}^3}{\mathbf{T}^2}}$$

$n = 1$ in vacuum, so consider $n - 1$.

variables	$\frac{e^2}{4\pi\varepsilon_0}$	m	N	ω	$n-1$
dimensions	$\frac{\mathbf{M}\,\mathbf{L}^3}{\mathbf{T}^2}$	$\mathbf{M}$	$\frac{\mathbf{1}}{\mathbf{L}^3}$	$\frac{\mathbf{1}}{\mathbf{T}}$	$\mathbf{1}$

Properties of material medium

N = number of electrons per volume

m = mass of an electron

Also

ω = some mysterious angular frequency (of what ?)

Dimensional analysis yields

$$n - 1 \sim \frac{e^2}{4\pi\varepsilon_0} \frac{N}{m\,\omega^2}$$

. . . but let us think more carefully about this $\frac{1}{\omega^2}$.

$$n - 1 \sim \frac{e^2}{4\pi\varepsilon_0}\,\frac{N}{m\,\omega^2}$$

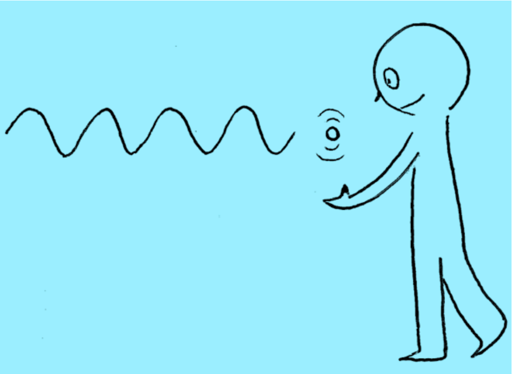
The electrons in the medium oscillate according to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,z + \mu\frac{\mathrm{d}}{\mathrm{d}t}\,z + {\omega_{\mathrm{res}}}^2\,z \propto \exp(\,i\omega t\,)$$

damping coefficient

resonant frequency
of the material

forcing by light



$$\Longrightarrow\; z(t) \propto \frac{1}{{\omega_{\mathrm{res}}}^2 - \omega^2 + i\,\mu\,\omega}\,\exp(\,i\omega t\,)$$

Hence the mysterious $\frac{1}{\omega^2}$ should in fact be $\frac{1}{{\omega_{\mathrm{res}}}^2 - \omega^2 + i\,\mu\,\omega}$.

A full calculation reveals

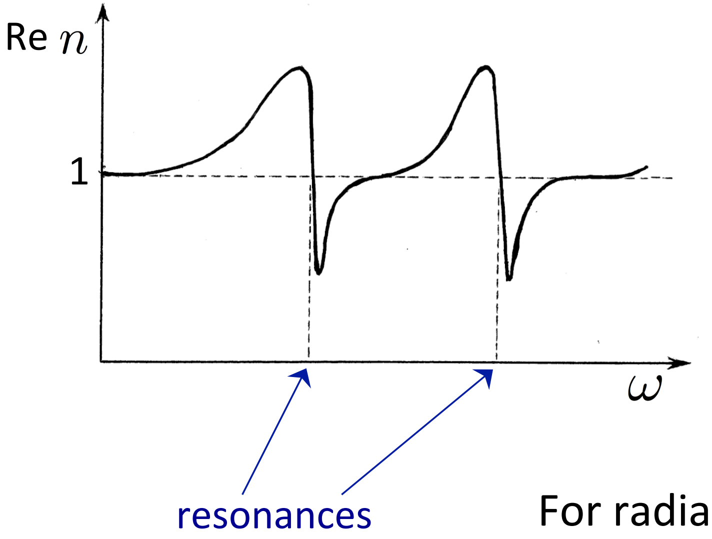
$$\boxed{\frac{n-1}{2\pi} = \frac{e^2}{4\pi\varepsilon_0\,m}\,\frac{N}{{\omega_{\mathrm{res}}}^2 - \omega^2 + i\,\mu\,\omega}}$$

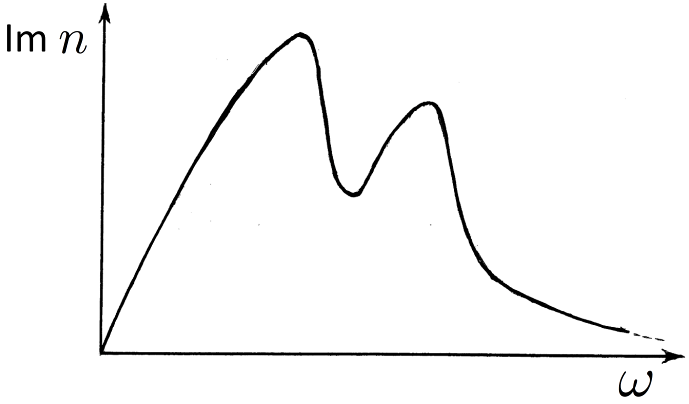
For material made of multiple species of atoms,

$$\boxed{\frac{n-1}{2\pi} = \frac{e^2}{4\pi\varepsilon_0\, m}\sum_s \frac{N_s}{\omega_{\mathrm{res}}(s)^2 - \omega^2 + i\,\mu_s\,\omega}}$$

Remarkable properties of n :

- $n-1 \propto N$
- n is *complex* !
- Re n is the refractive index proper, while Im n is the ***absorptive index*** .
- The graphs of Re n and Im n look like these :





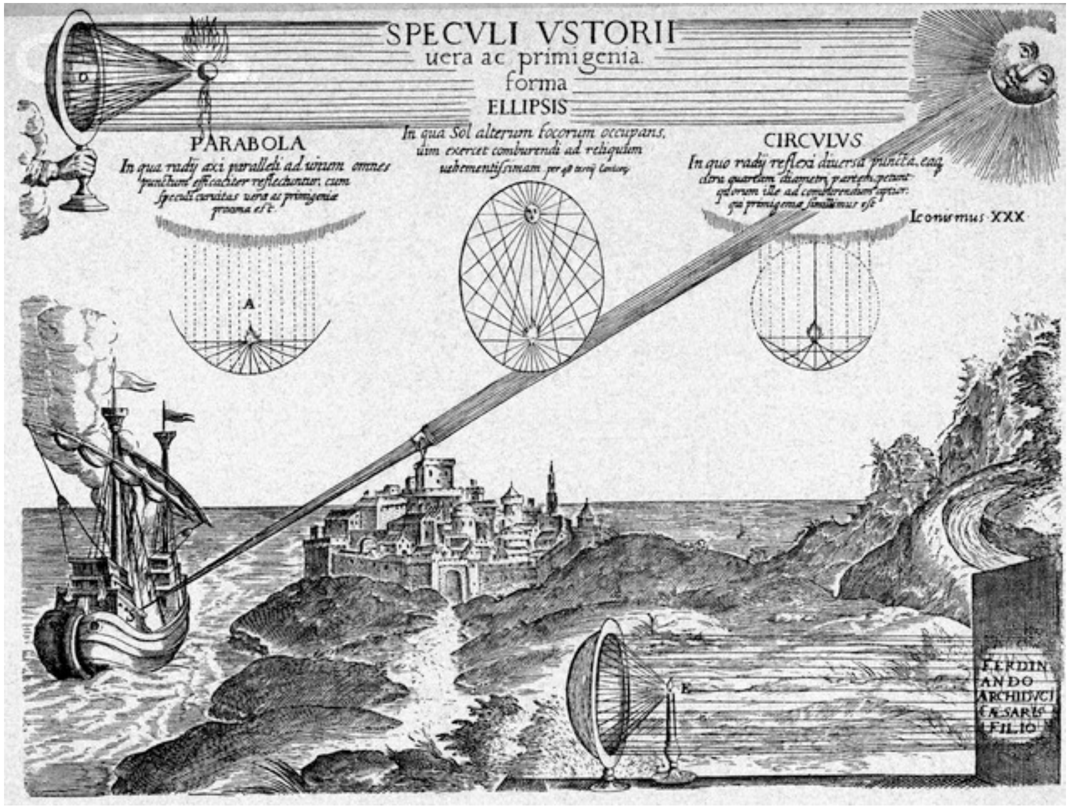
For radiations and optics, see

Feynman Lectures on Physics, vol.1 (Addison Wesley)

Wherever wave phenomena occur,
there are applications of refraction/absorption
(optics, crystallography, acoustics, seismology, tsunami defocusing. . .) .

We will see some of these, and assume for a while that η is real.

Note also :
reflection can be treated as refraction with $\eta < 0$.



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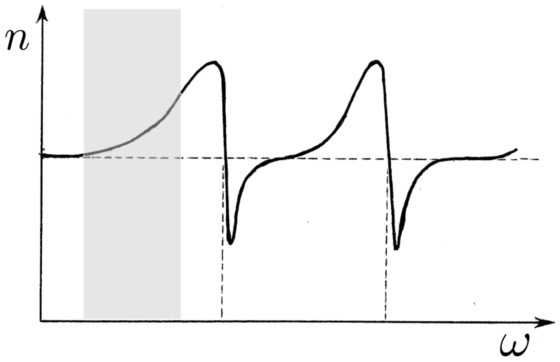
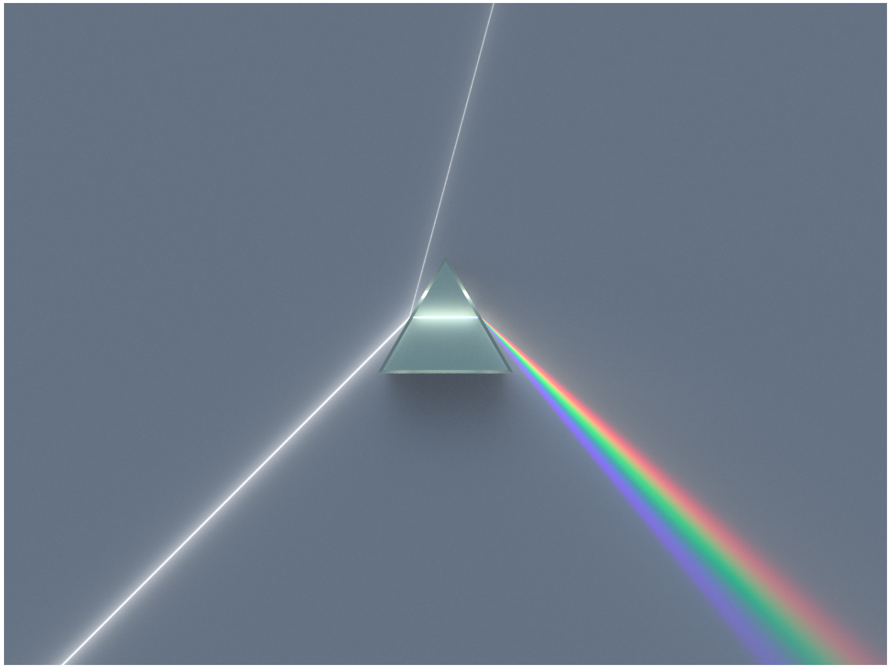
Intermission

For most materials, $\omega_{\rm res}$ is in the ultraviolet $\gg \omega$ of visible light.



$\Longrightarrow n$ increasing function of ω

$n_{\text{blue}} > n_{\text{red}}$

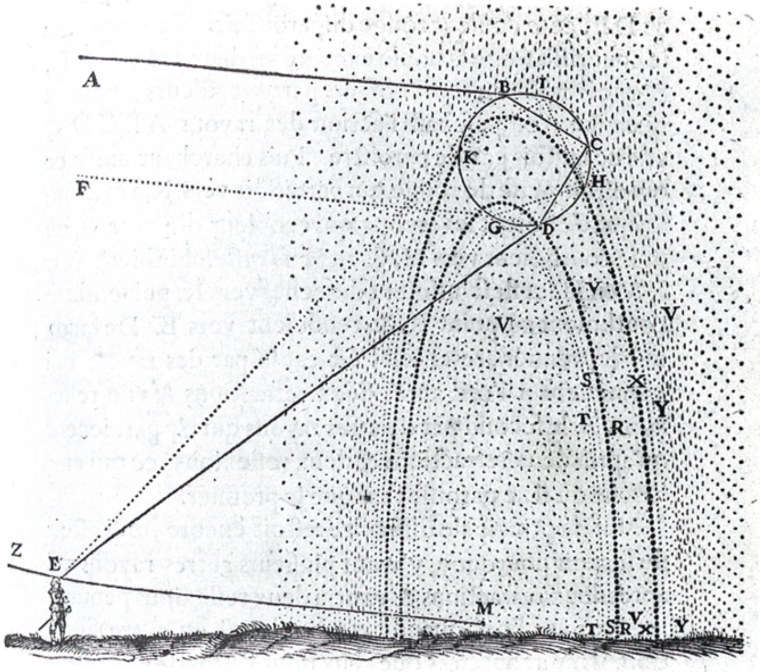


A prism refracts **blue** more than **red**.

Rainbow

Water droplets in the atmosphere act as prisms.

As sunlight enters a droplet it gets refracted,
reflected internally,
and refracted again as it exits the droplet.



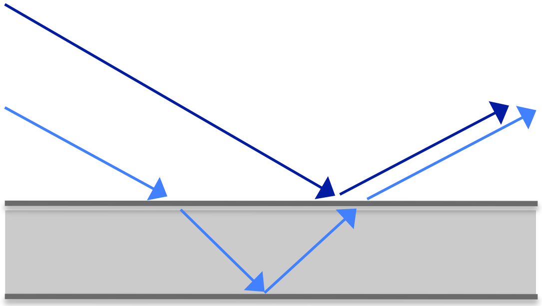
[Descartes 1637]

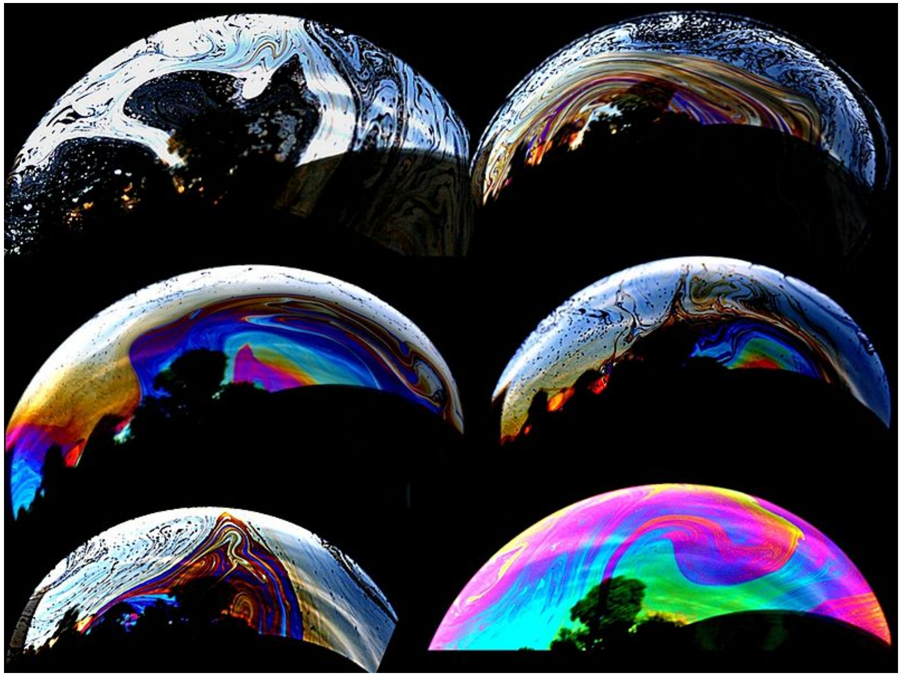
Depending on whether light got *reflected once, twice*, . . .
before exiting the droplets,
we have *primary, secondary*, . . . rainbows.

Observe the order of the colors.

Soap film

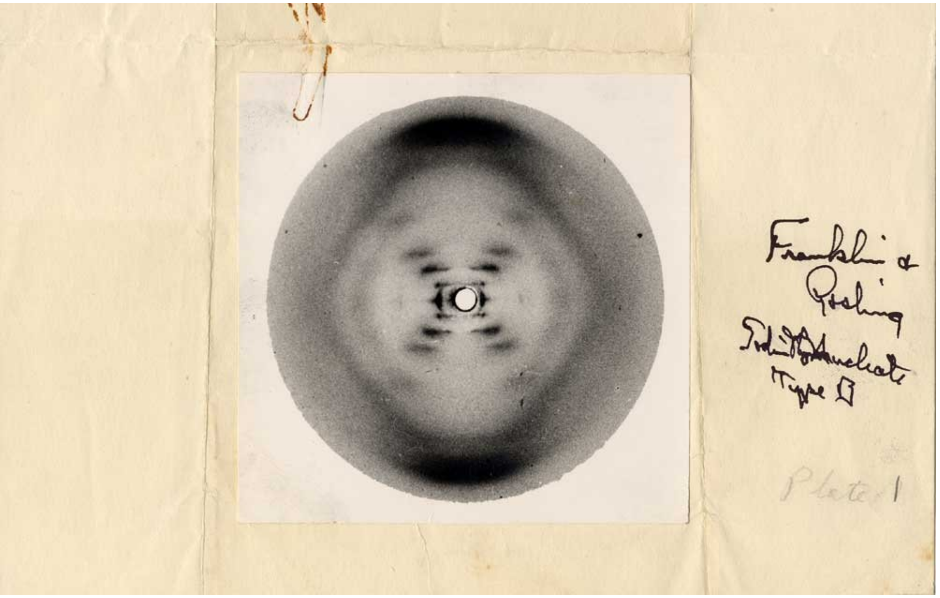
In a thin layer of soap, light gets refracted,
reflected internally,
and refracted again as it exits the film
but this time the exiting light ***interferes*** also with the light *reflecting externally*.





Discovery of DNA double helix

Refraction and interference occur with any frequencies of electromagnetic waves, e.g. with X-ray (but we speak of ***diffraction***).

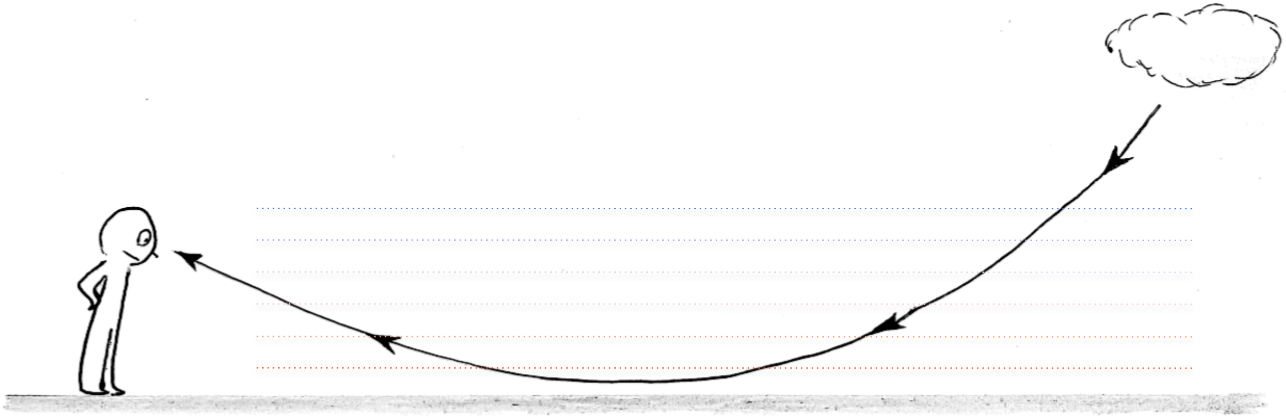


[Franklin and Gosling 1951]

A `rainbow’ from a DNA led to the determination of its structure.

Mirage

$$n - 1 \propto N$$

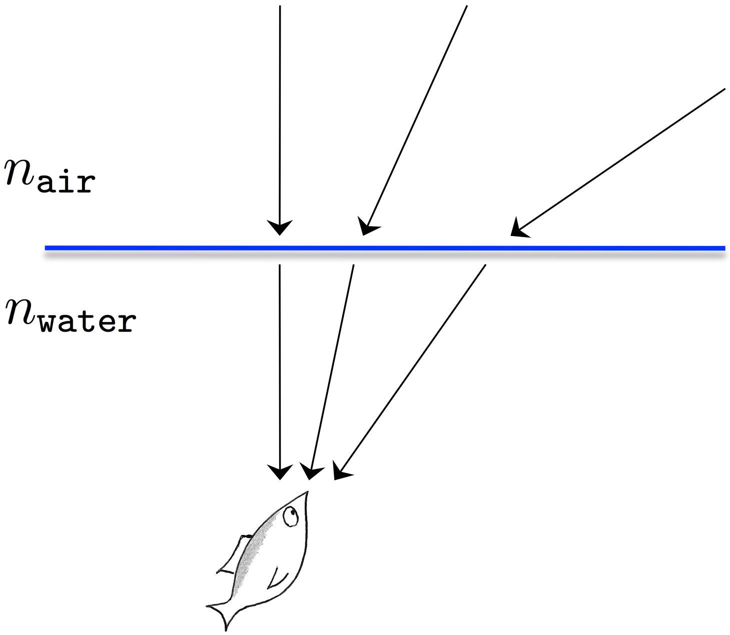


When air cools,
its density N increases.

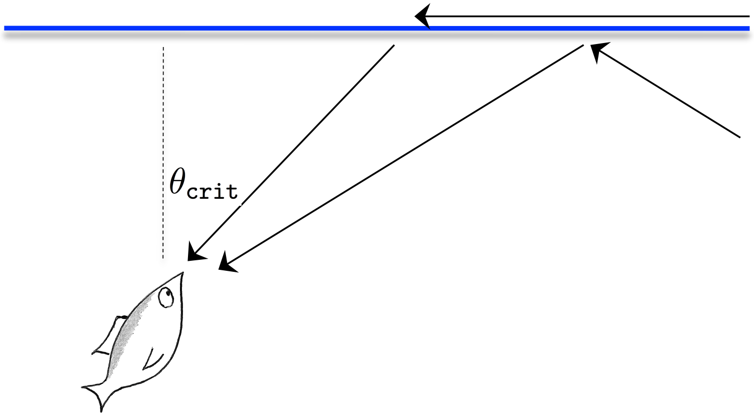
$$\Longrightarrow \; n_{\text{cool}} > n_{\text{warm}}$$

Below a cooler air and
above a warmer ground,
we have effectively
a continuous stratification of prisms.

Sky seen from under water



The whole sky is shrunk to a `porthole' subtending a cone of half-angle $\theta_{\texttt{crit}}$



$$n_{\texttt{air}} \sin\frac{\pi}{2} = n_{\texttt{water}} \sin\theta_{\texttt{crit}}$$

Outside this cone, you see reflections of the underwater landscape.

With

$$n_{\texttt{air}} \approx 1.0003$$
$$n_{\texttt{water}} \approx 1.33$$

we find

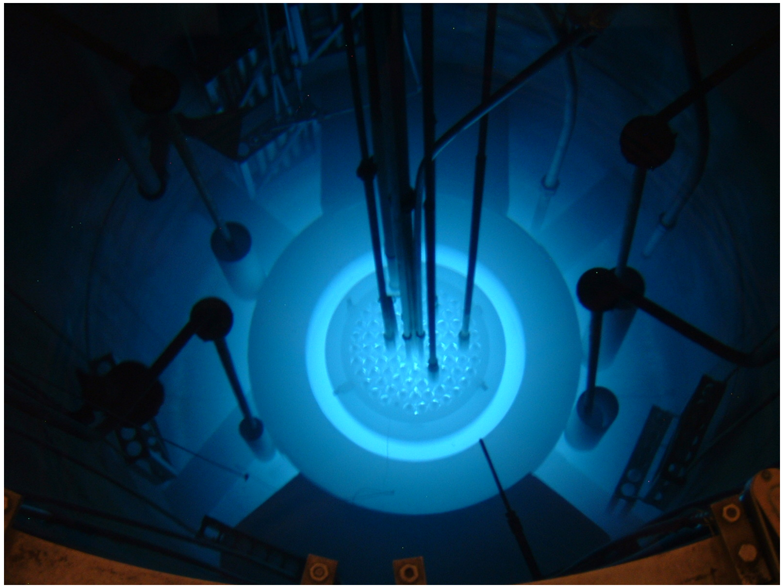
$$n_{\texttt{air}} \sin \frac{\pi}{2} = n_{\texttt{water}} \sin \theta_{\texttt{crit}}$$

$$\Longrightarrow \boxed{\theta_{\texttt{crit}} \approx 48^\circ}$$

Čerenkov radiation

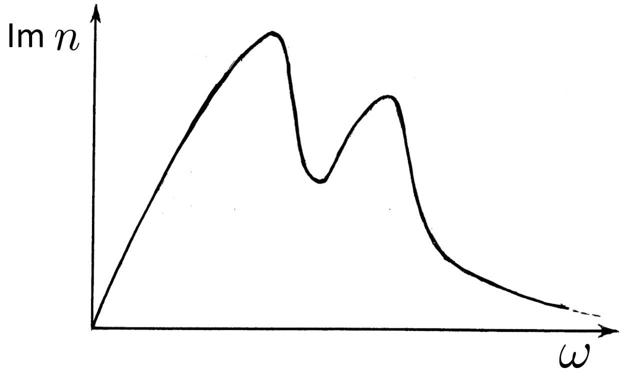
When a charge travels through a material *faster than the speed of light*
. . . in that material (possible if $n > 1$!) , a kind of shock wave is observed.

[1958 Nobel prize to Čerenkov, Frank, Tamm]



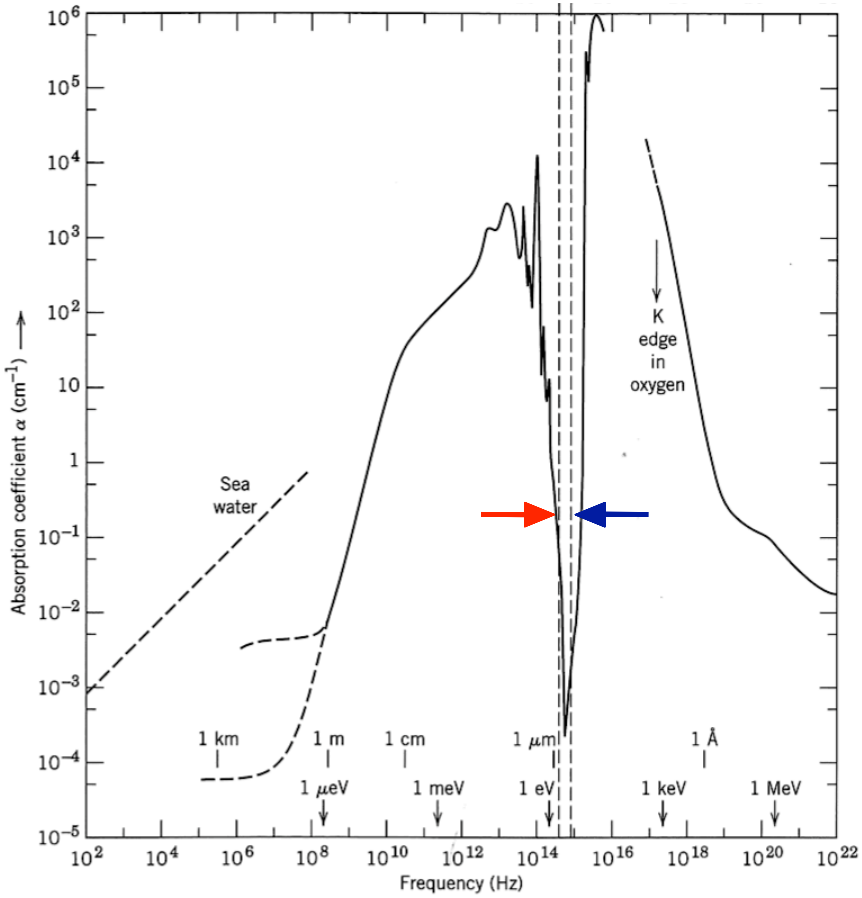
A blue glow in a pool of a nuclear reactor is an example,
due to fast charges emitted by fission.

For our final example, remember that the index n is *complex* and that Im n represents the absorptive index.



Graph of the absorptive index of water

It has a narrow window → ← in which Im n plunges by factor of 10^8, i.e. water passes electromagnetic waves at these frequencies but blocks them off at all others.

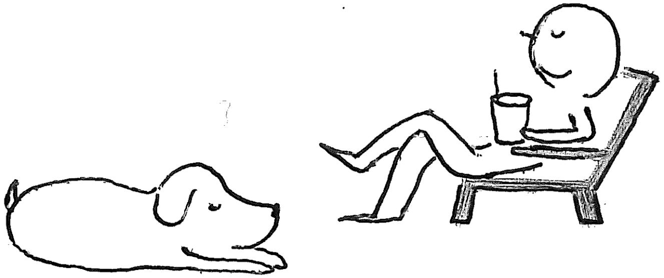


This window is exactly the \`visible' spectrum.

$\Longrightarrow$ *evolutionary* significance

Review of what we saw in lecture 3/3

- sound of a splash
- Snell's law
- estimating the refractive index
- refractive index is complex, reflection is negative refraction
- rainbow, soap film, DNA, mirage, sky from under water, Čerenkov
- H_2O and evolution of vision



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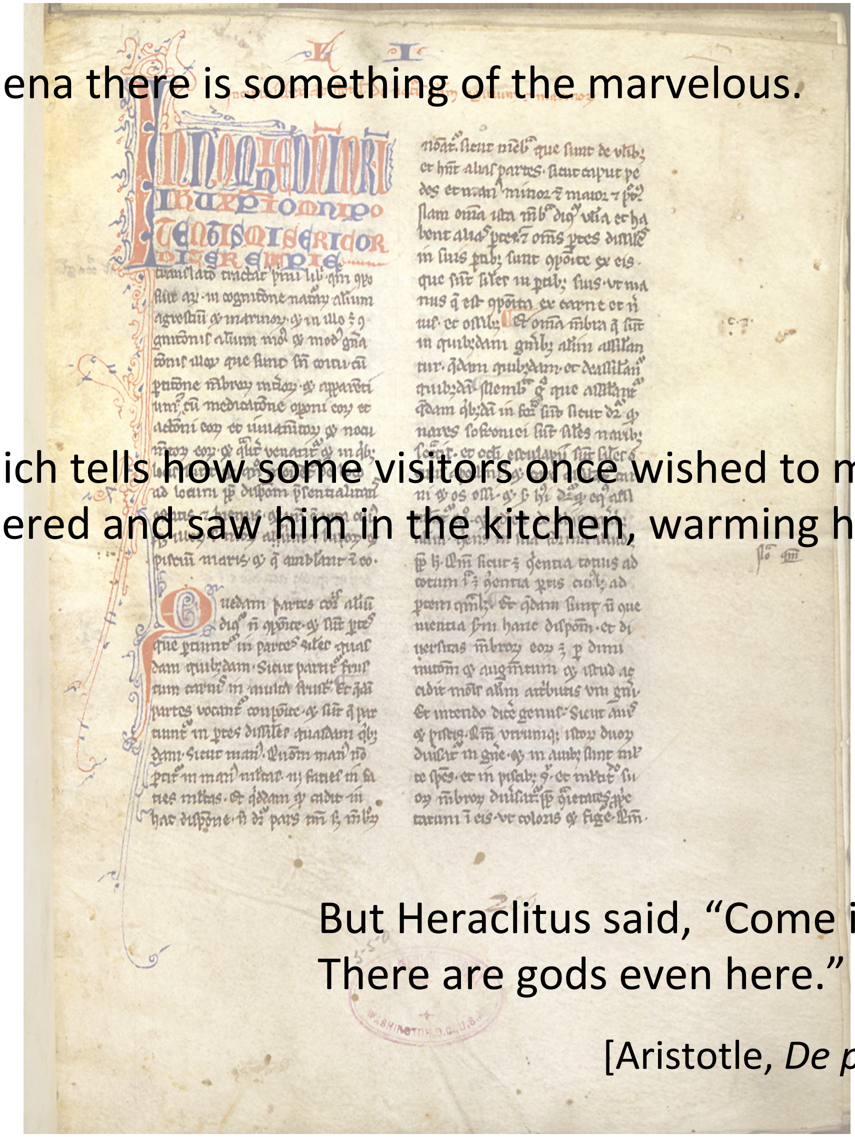
A Message



In all natural phenomena there is something of the marvelous.

There is a story which tells how some visitors once wished to meet Heraclitus, and when they entered and saw him in the kitchen, warming himself at the stove, they hesitated.

But Heraclitus said, “Come in; don’t be afraid.
There are gods even here.”

[Aristotle, *De partibus animalium*]

Don’t be afraid,
try simple modeling of your own phenomena.

There is mathematics everywhere.

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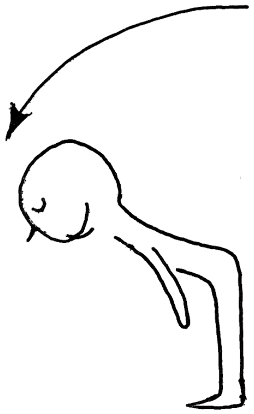

Thanks for your attention