Partitions

(1) Use Euler's recurrence for the partition function $p(n)$ to prove that there are infinitely many n (resp. m) for which $p(n)$ (resp. $p(m)$) is even (resp. odd).

(2) Use Jacobi's Triple Product Identity to prove that

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{(k^2+k)/2}.$$

(3) Recall that Ramanujan proved that

$$p(5n+4) \equiv 0 \pmod 5,$$

$$p(7n+5) \equiv 0 \pmod 7,$$

$$p(11n+6) \equiv 0 \pmod{11}.$$

a) If ℓ is prime, then define integers $a_\ell(n)$ by

$$\sum_{n=0}^{\infty} a_\ell(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^{\ell-1}.$$

For $0 \leq \delta < \ell$, prove that

$$p(\ell n + \delta) \equiv 0 \pmod{\ell}$$

for all $n \geq 0$ if and only if

$$a_\ell(\ell n + \delta) \equiv 0 \pmod{\ell}.$$

(Hint. Use the infinite product generating function for $p(n)$ and the fact that $(1-X)^\ell \equiv 1 - X^\ell \pmod{\ell}$.)
b) For $\ell = 5$, give a closed formula for $a_5(n)$ using the fact that

$$\sum_{n=0} a_5(n)q^n = \left(\prod_{n=1}^{\infty}(1-q^n)\right)\cdot\left(\prod_{n=1}^{\infty}(1-q^n)^3\right).$$

(Hint. Use Euler's and Jacobi's identities.)
c) Prove that $a_5(5n+4) \equiv 0 \pmod 5$ for all $n \geq 0$.

(4) Using the method above, prove that $p(7n+5) \equiv 0 \pmod 7$ for all $n \geq 0$.

(5) Try to prove that $p(11n+6) \equiv 0 \pmod{11}$ for all $n \geq 0$.