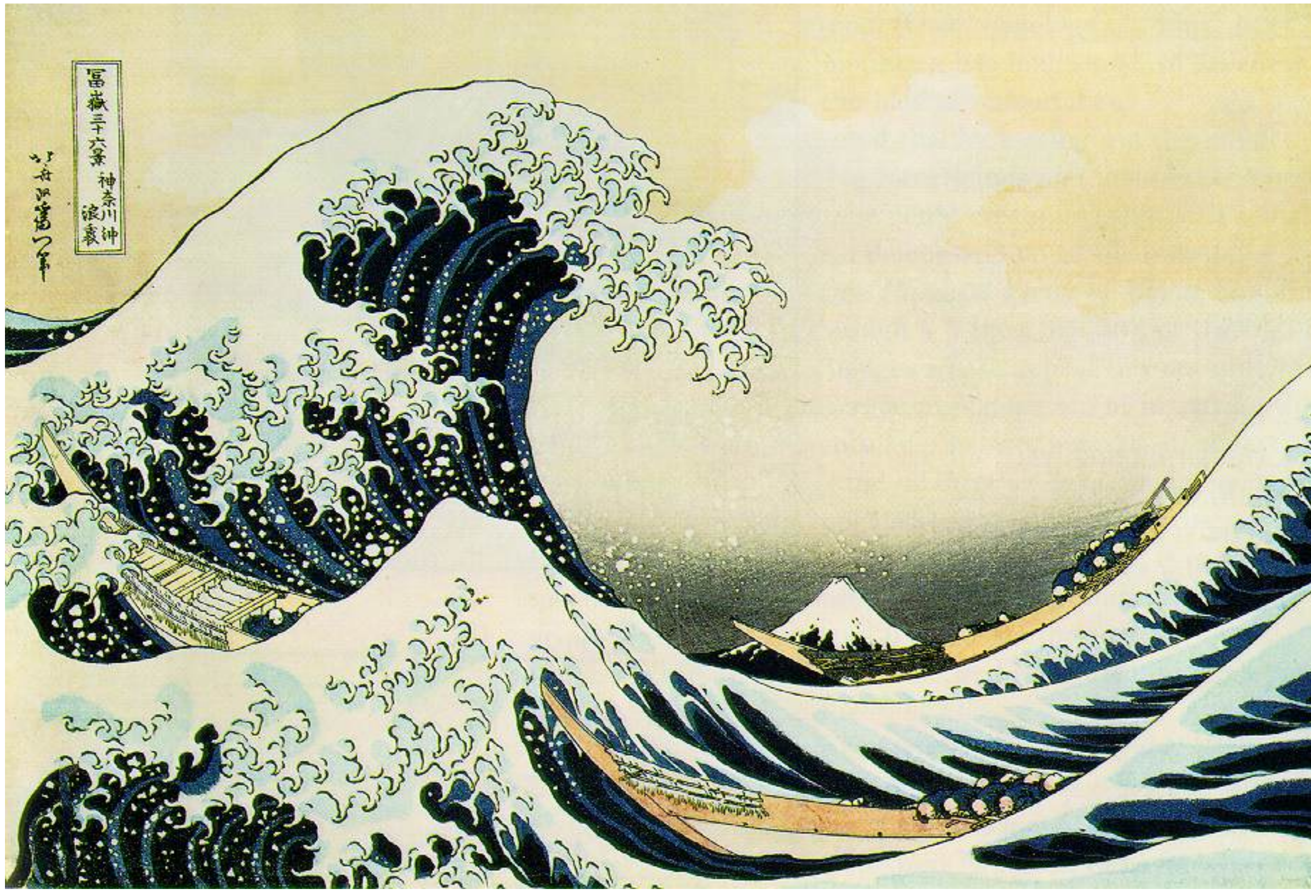
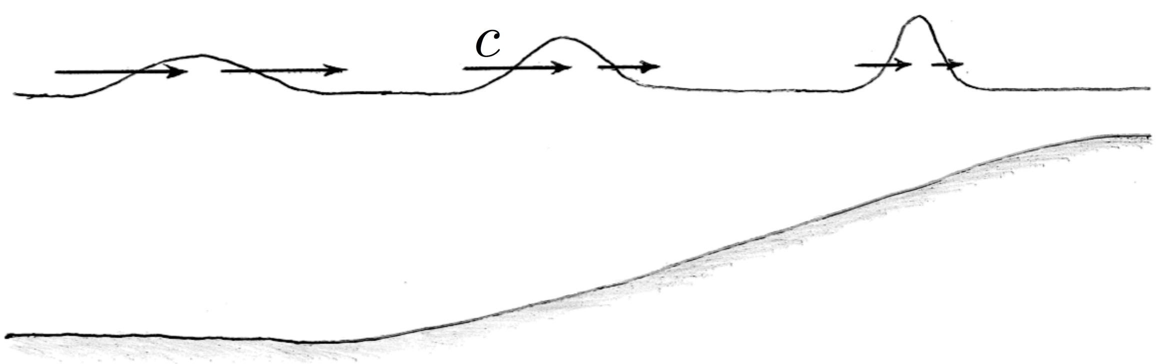
An invitation to simple modeling of complex phenomena

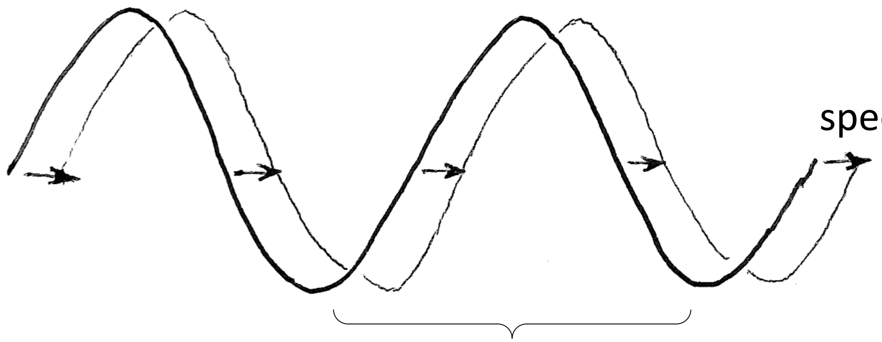


T. Tokieda
Lyon, August 2012

What determines the ***speed*** c of a wave ?



Variables governing a wave :

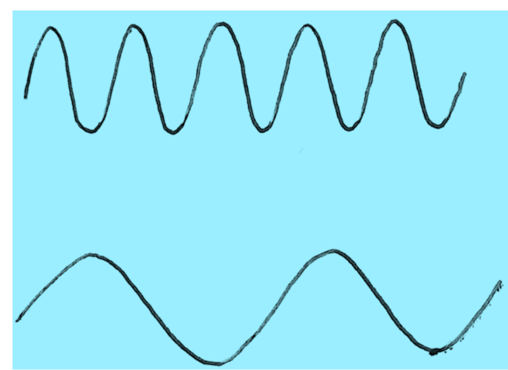


angular frequency

$$\omega = \frac{2\pi}{\text{period}}\,, \quad [\,\omega\,] = \frac{\mathbf{1}}{\mathbf{T}}$$

wavenumber

$$k = \frac{2\pi}{\text{wavelength}}\,, \quad [\,k\,] = \frac{\mathbf{1}}{\mathbf{L}}$$



speed

$$c = \frac{\omega}{k}\,, \quad [\,c\,] = \frac{\mathbf{L}}{\mathbf{T}}$$

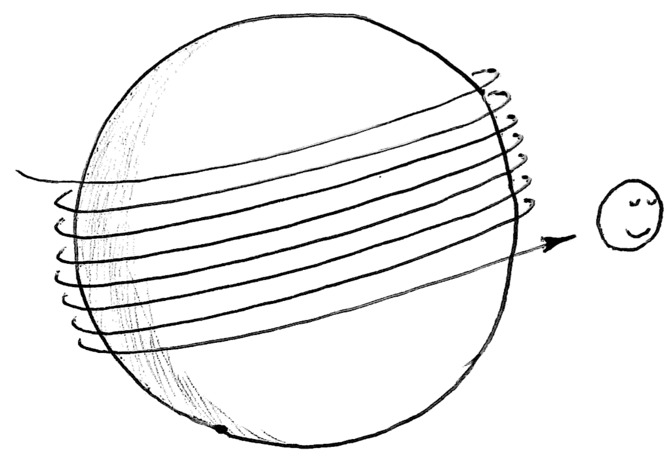
Once we know how ω and k are related, we have c !

$\omega = \omega(k)$ is called the ***dispersion relation*** for the given wave.

For example, in case of light,

$$\omega = 299792458\ldots(\,\mathrm{m/sec}\,)\cdot k \qquad \text{(linear relation)}$$

$$\Longrightarrow\ c = \frac{\omega}{k} = \mathrm{const.}$$



7.5 tours of the Earth per sec

Dispersion relation for water wave :

1) *Dimensional analysis*

We think about physics: hmm. . .

Without gravity, crests of the wave would continue rising and would not fall back down. So let us include g :

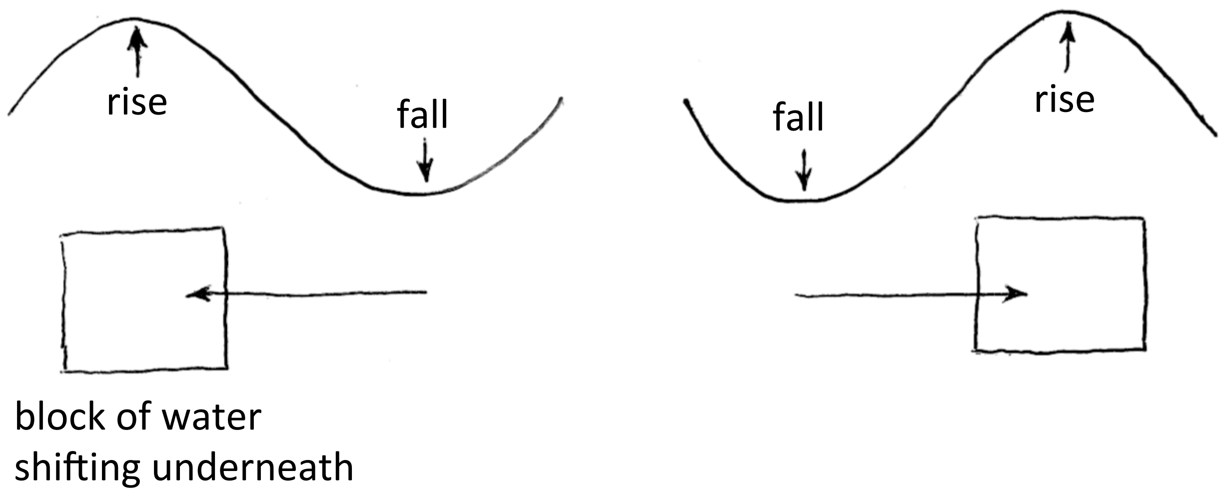
variables	k	g	$\cancel{\rho}$	ω
dimensions	$\dfrac{\mathbf{1}}{\mathbf{L}}$	$\dfrac{\mathbf{L}}{\mathbf{T}^2}$	$\cancel{\dfrac{\mathbf{M}}{\mathbf{L}^3}}$	$\dfrac{\mathbf{1}}{\mathbf{T}}$

ω cannot depend on ρ , since no other variable cancels $\mathbf{M}$.

$$\boxed{\omega \sim \sqrt{gk}\,, \quad c = \frac{\omega}{k} \sim \sqrt{\frac{g}{k}}}$$

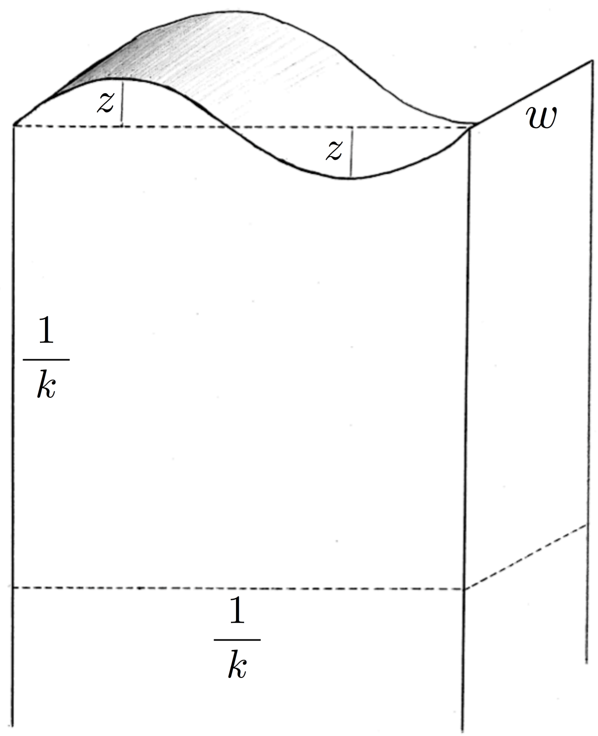
2) *Back-of-the-envelope estimate*

A cartoon model of how a wave works :



Periodic oscillation near equilibrium

. . . model it as a ***spring*** !



force = surface weight difference

$$\approx \rho \cdot \left(z \, \frac{1}{k} \, w \right) \cdot g$$

mass of shifted block

$$\approx \rho \cdot \left(\frac{1}{k} \, \frac{1}{k} \, w \right)$$

$\Longrightarrow$ acceleration $= \dfrac{\text{force}}{\text{mass}} \approx z\, g\, k$

Also, the surface oscillates with acceleration $\approx \omega^2 z$

Identifying these accelerations,

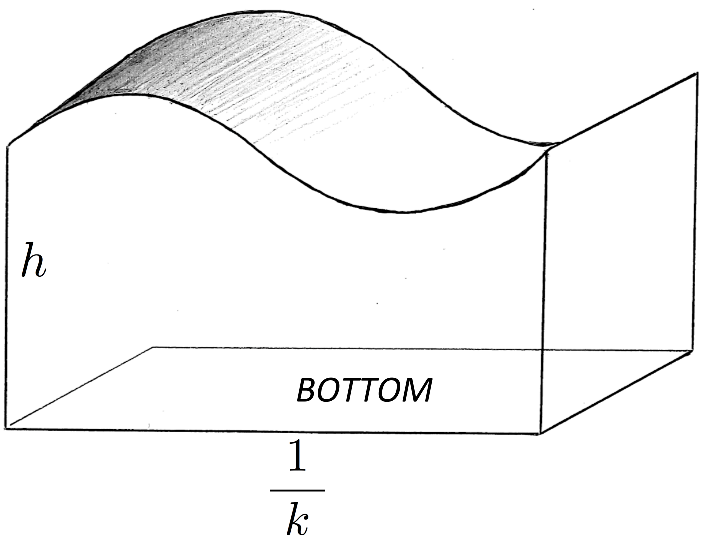
$$\boxed{\omega \approx \sqrt{gk}}$$

. . . we have improved $\sim$ to $\approx$.

3) *Solving the full equation*

For the full equation, see

Faber, *Fluid Dynamics for Physicists* (Cambridge UP)



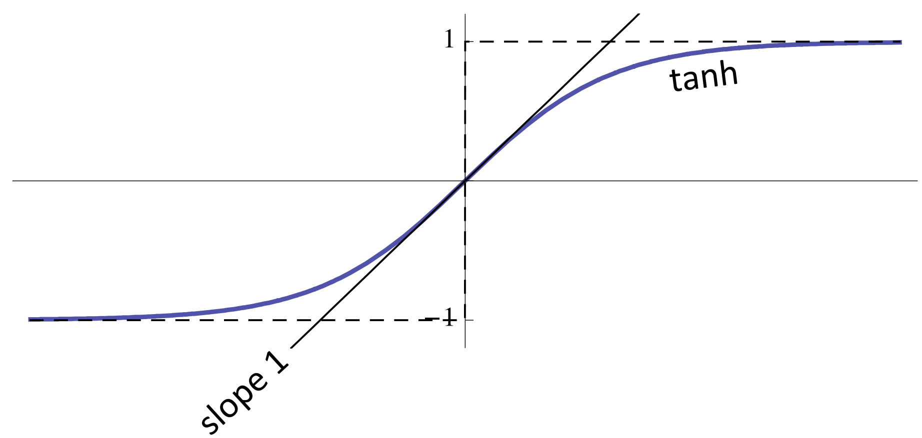
It allows us to account for the effect of finite depth h and gives

$$\omega = \sqrt{gk \tanh(hk)}$$

$$\approx \begin{cases} \sqrt{gk} & \text{deep water} \quad h \gg \dfrac{1}{k} \\ \sqrt{gk^2 h} & \text{shallow water} \quad h \leqslant \dfrac{1}{k} \end{cases}$$

So

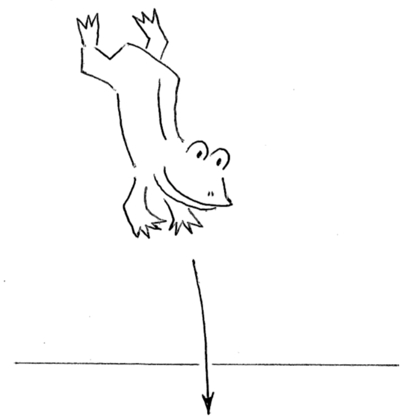
$$\boxed{c = \frac{\omega}{k} \approx \begin{cases} \sqrt{\dfrac{g}{k}} & \text{deep} \\ \sqrt{gh} & \text{shallow} \end{cases}}$$

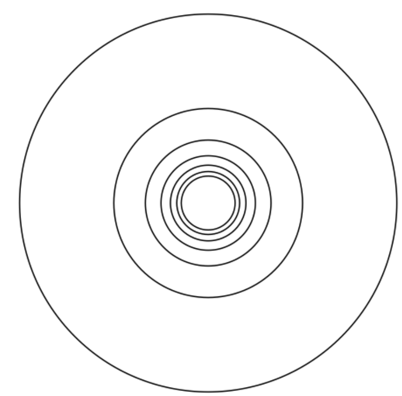


Application of $c \approx \sqrt{\frac{g}{k}}$ (depth $\gg$ wavelength)

In this regime, *long waves travel faster than short waves.*

古池や
蛙飛こむ
水の音

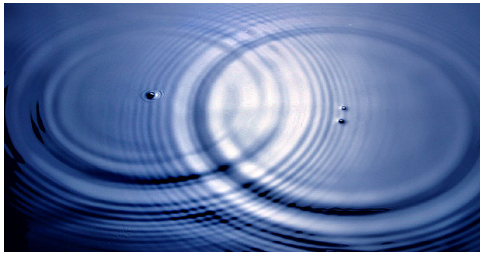




Small k have already reached outside while large k are still lagging inside.

Aside :

We have been neglecting the effect of surface tension σ , $[\,\sigma\,] = \dfrac{\text{energy}}{\text{area}} = \dfrac{\mathbf{M}}{\mathbf{T}^2}$.

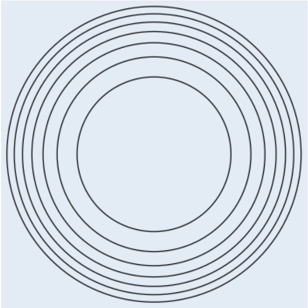


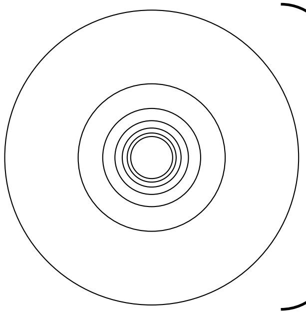
For *capillary waves* (*ripples*) driven by surface tension, the dispersion relation becomes

hence

$$\omega = \sqrt{\frac{\sigma}{\rho}k^3}$$

$$c = \sqrt{\frac{\sigma}{\rho}k}$$



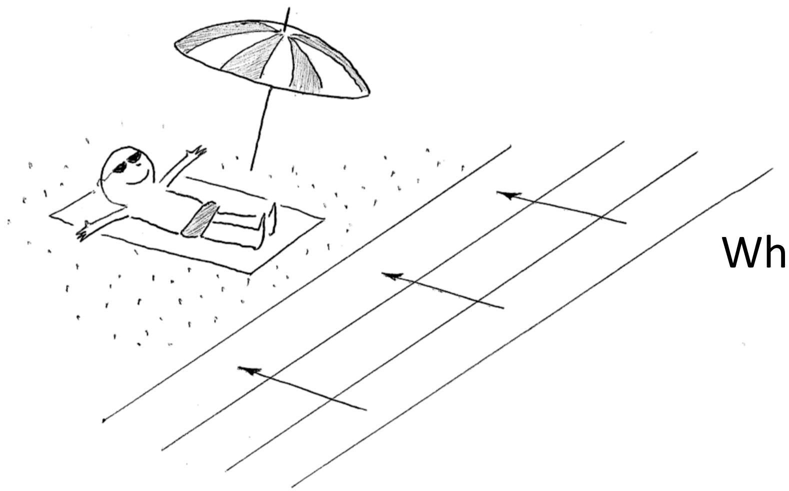
(compare )

i.e. *short waves travel faster than long waves*.

The most general dispersion relation turns out to be

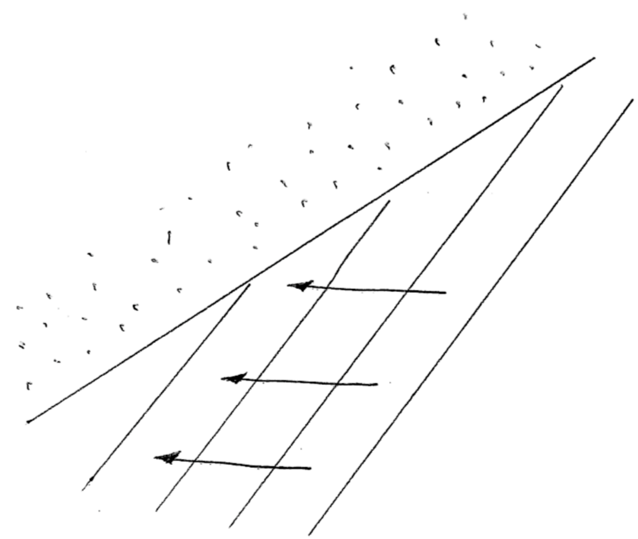
$$\boxed{\omega = \sqrt{\left(gk + \frac{\sigma}{\rho}k^3\right)\tanh(hk)}}$$

Applications of $c \approx \sqrt{gh}$ (depth $\leqslant$ wavelength)

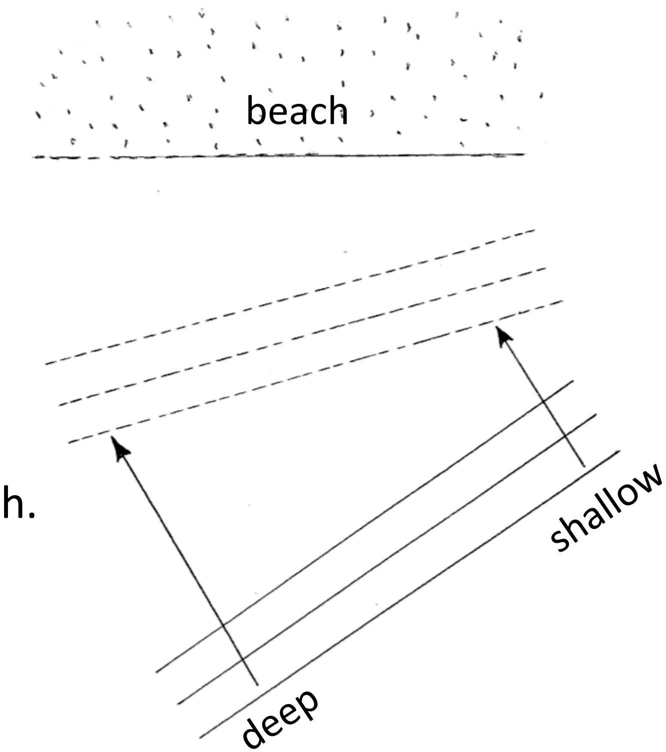


Why do waves arrive ***parallel*** to the beach

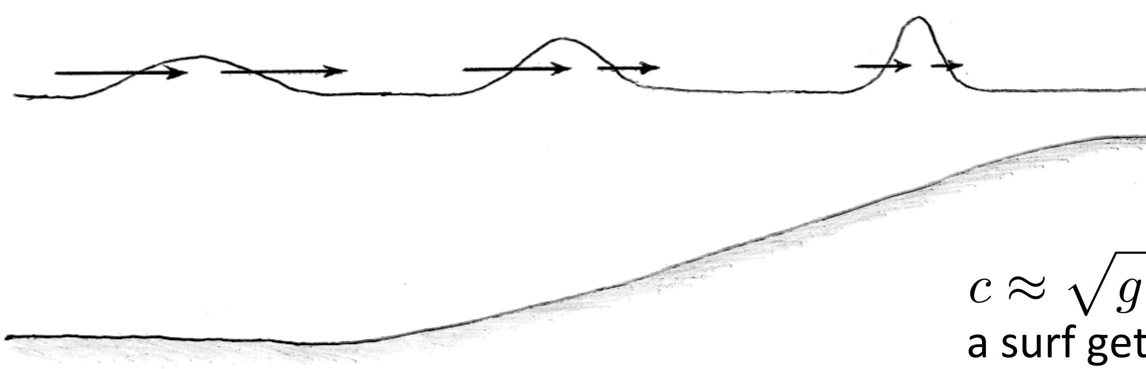
. . . and rarely at an angle like this ?



The reason is $c \approx \sqrt{gh}$.



As a wave comes in, its front turns,
to become more and more parallel to the beach.



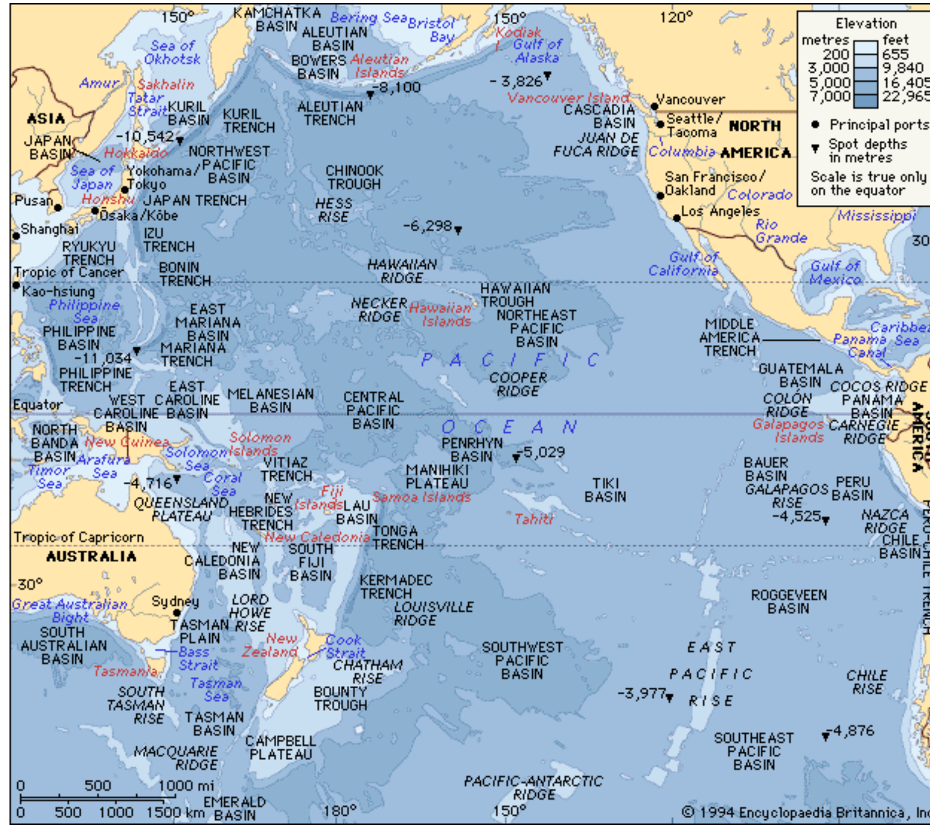
$c \approx \sqrt{gh}$ also explains why
a surf gets compressed and stands up.

How quickly does a ***tsunami*** cross the Pacific ?

distance Tokyo $\longleftrightarrow$ Santiago (Chile)

$$\approx 17000\,\mathrm{km}$$

(almost half way around the Earth)



average depth $h \approx 4 \times 10^3\,\mathrm{m}$

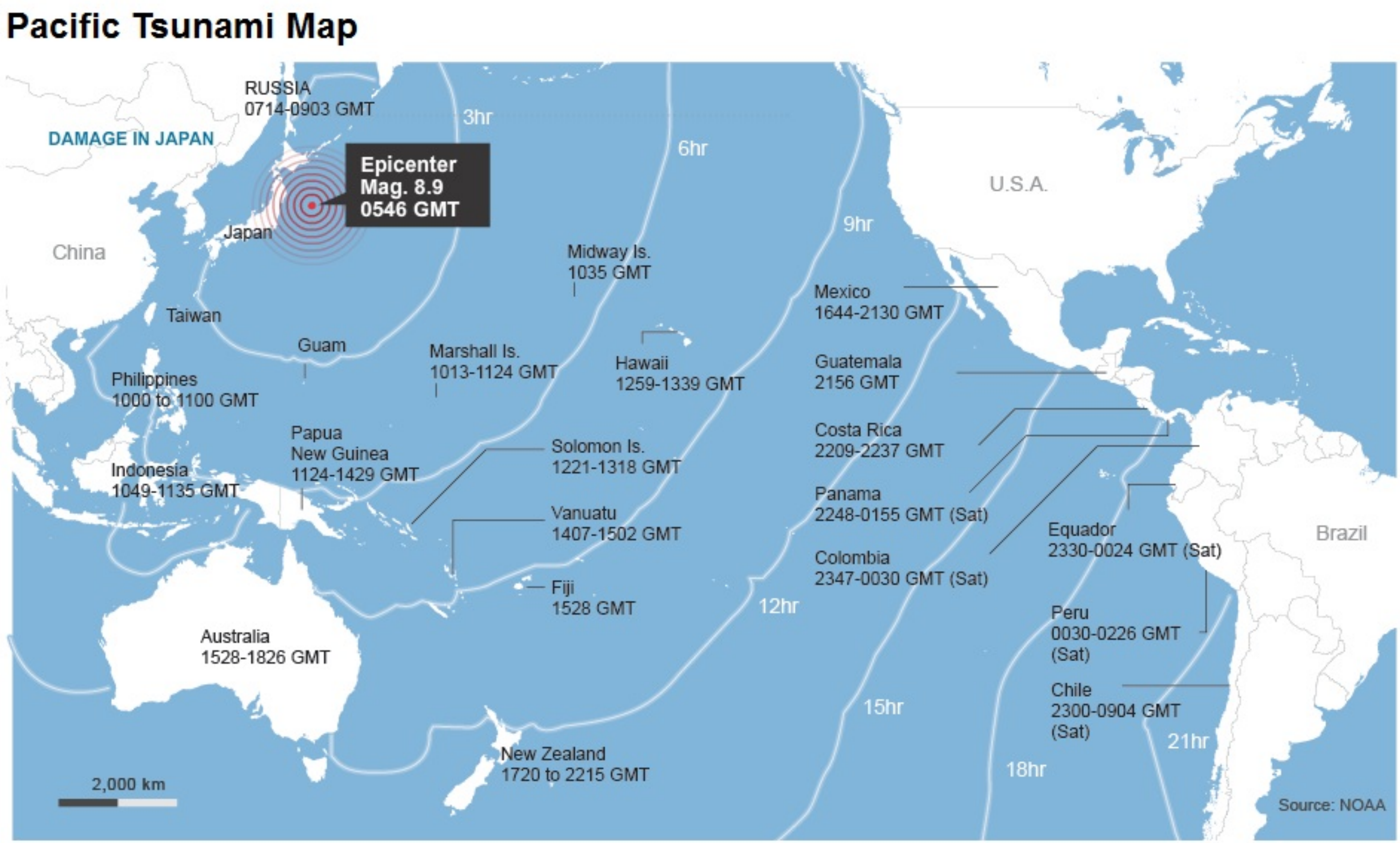
and we know $g \approx 10\,\mathrm{m/sec^2}$

$$\Longrightarrow\ c \approx \sqrt{gh} \approx 200\,\mathrm{m/sec} = 720\,\mathrm{km/hour}$$

(a bit slower than a jumbo jet)

$$\Longrightarrow\ \frac{17000}{720} \approx \boxed{24\ \mathrm{hours}}$$

Observed data confirm this estimate :



The success of our shallow-water model suggests that
for a tsunami, *even the Pacific Ocean is `shallow'*.
Indeed, a tsunami's wavelength mid-ocean is typically tens of km $\geqslant$ depth 4km.

なかやすみ

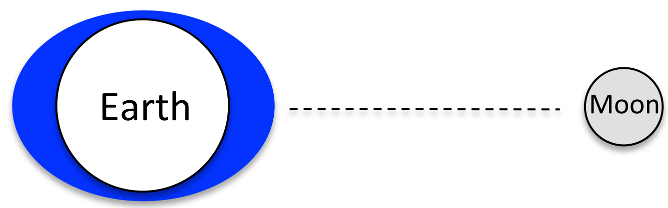
Intermission

We have been studying free oscillators
but we have not yet considered ***external forcing*** .

We now consider phenomena involving forcing (tide, pendulum).

Another surprising application of $c \approx \sqrt{gh}$

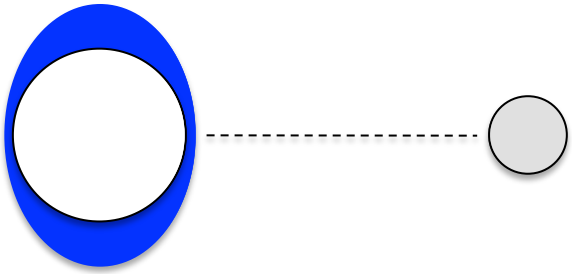
Every textbook shows the picture of ***tide*** like this :



Tide is an effect of the *gradient* of gravitation, and it is true that the Moon's gravitation induces **2 *bulges*** of water on the Earth.

But the *orientation* of the bulges shown is incorrect.

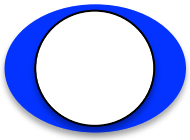
The correct orientation turns out to be this :



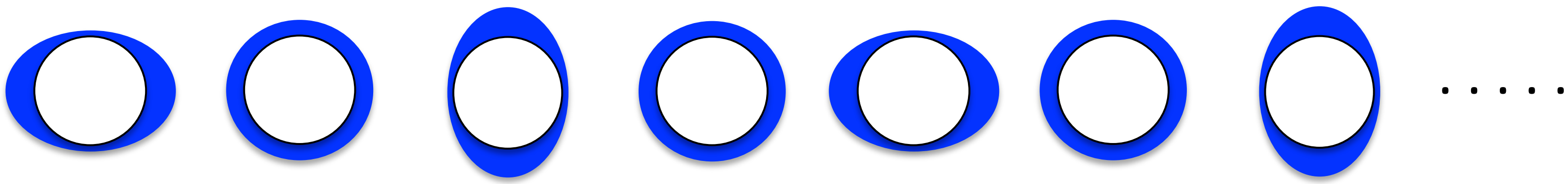
Why ?

The water on the Earth is an oscillator, again like a ***spring*** .

At first, *suppose no Moon*.
If the water is released from the initial configuration



then it will oscillate *freely*



Period of this ***free oscillation*** $=$ time for a tsunami to go half way around the Earth

$$\approx 24 \text{ hours}$$

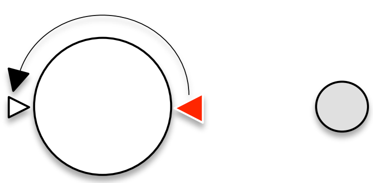
Now *bring back the Moon*.
Then the water becomes a ***forced oscillator*** , with

period of forcing

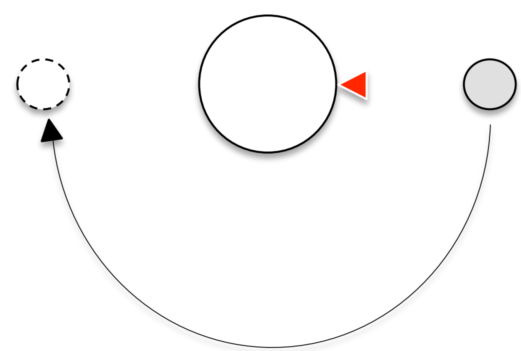
$=$ time for the Earth to rotate so as to position the Moon on the opposite side

$\approx 12\,\mathrm{hours}$

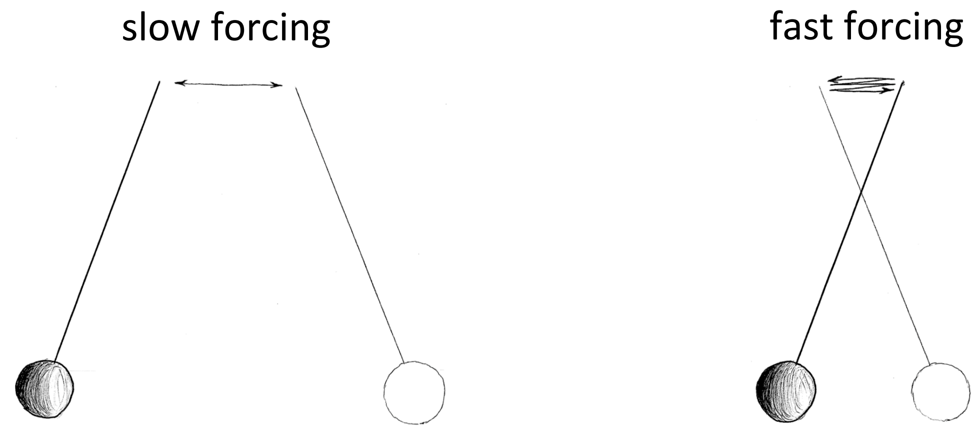
inertial view



view from an observer fixed on the Earth



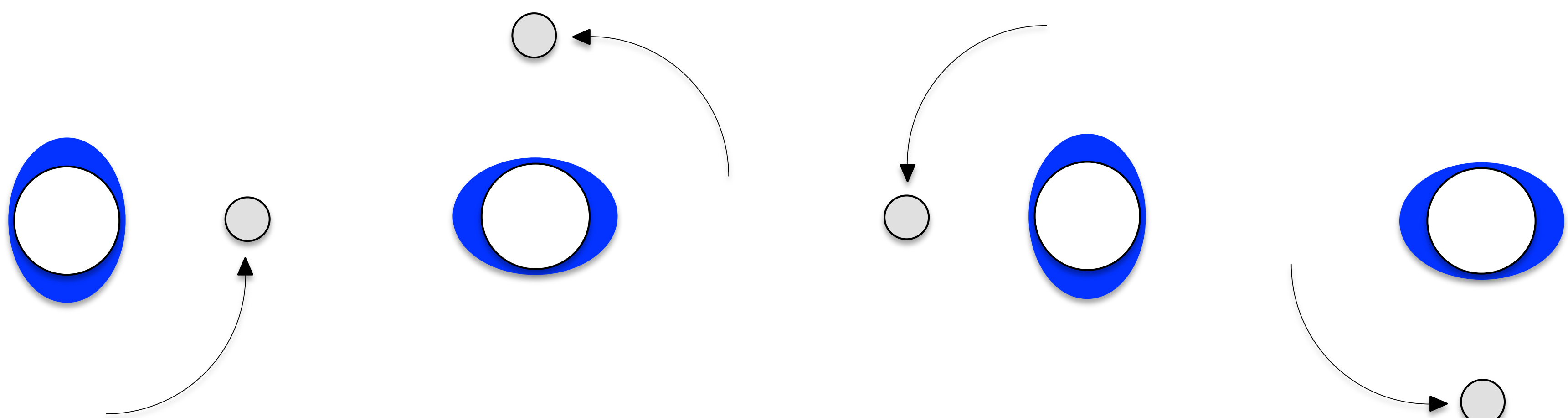
An oscillator responds to periodic forcing in 2 different ways :



oscillates ***in phase*** with forcing

out of phase

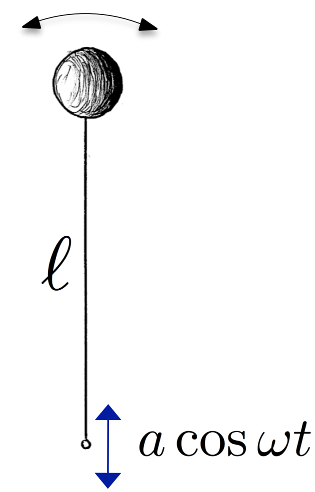
Since $12 < 24$, the Moon's forcing is *fast* ,
so the water responds *out of phase* , at 90° to the position of the Moon.



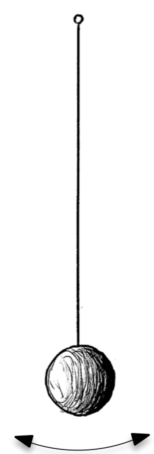
Upside uʍop pendulum

[Kapitsa 1951]

Periodic forcing has other curious effects.



For an ordinary pendulum,
the ***stable equilibrium*** is
the downward position.



But if the pivot is shaken fast enough

$$a\omega > \sqrt{2g\ell}$$

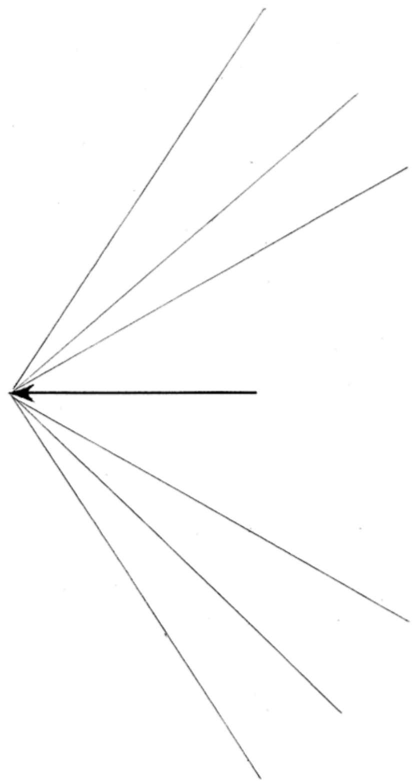
the upward position becomes stable.

Kelvin wedge

As a duck swims, it leaves in its wake a wedge-shaped pattern.

What is the ***angle*** of this wedge ?

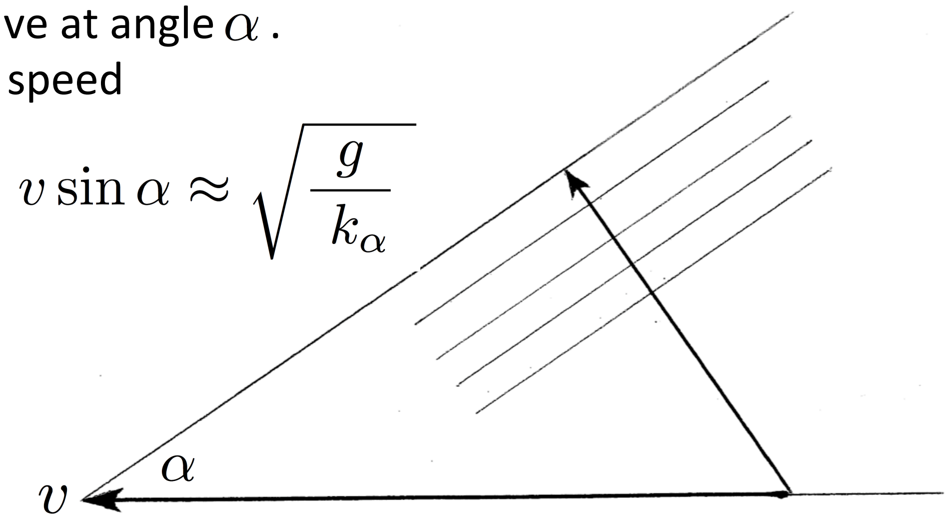
Answer : about 39°, *independent of the duck's speed or size*.



The duck, advancing at speed v ,
excites waves that spread at various angles.

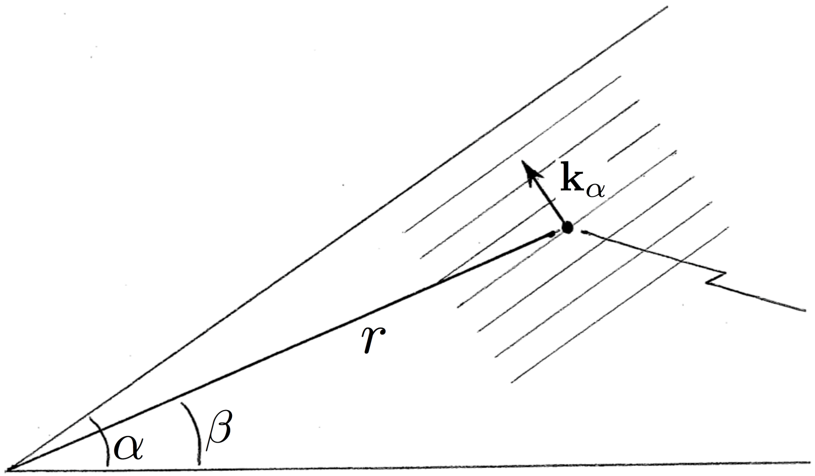
Consider the wave at angle α .
It propagates at speed

$$v \sin\alpha \approx \sqrt{\frac{g}{k_\alpha}}$$



$$\Longrightarrow \; -\frac{\mathrm{d}k_\alpha}{\mathrm{d}\alpha} \approx \frac{2k_\alpha}{\tan\alpha}$$

log derivative



$$-\frac{\mathrm{d}k_\alpha}{\mathrm{d}\alpha} \approx \frac{2k_\alpha}{\tan\alpha}$$

At a point of polar coordinates r , β the α wave's contribution is

$$\propto \exp(\, i\,\mathbf{k}_\alpha \cdot \mathbf{r}\,)$$

$$\mathbf{k}_\alpha \cdot \mathbf{r} = -k_\alpha r \sin(\alpha - \beta)$$

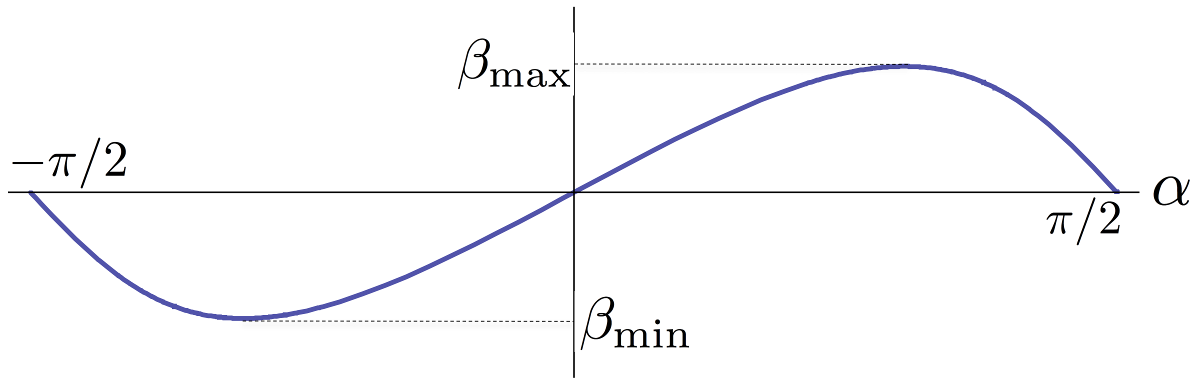
In the oscillatory sum $\int_{-\pi/2}^{\pi/2} \exp(\, i\,\mathbf{k}_\alpha \cdot \mathbf{r}\,)\,\mathrm{d}\alpha$ contributions from various α cancel,

except in a narrow range where $\mathbf{k}_\alpha \cdot \mathbf{r}$ varies slowly : ***method of stationary phase*** .

$$0 \approx \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathbf{k}_\alpha \cdot \mathbf{r} = -\frac{\mathrm{d}k_\alpha}{\mathrm{d}\alpha}\, r\sin(\alpha-\beta) - k_\alpha r\cos(\alpha-\beta)$$

$$\Longrightarrow\ 0 \approx \frac{2\tan(\alpha-\beta)}{\tan\alpha} - 1 \ \Longrightarrow\ \tan\beta \approx \frac{\tan\alpha}{2+\tan^2\alpha}$$

Graph of the *stationary-phase condition* $\tan\beta = \dfrac{\tan\alpha}{2+\tan^2\alpha}$



β has max, min at $\pm$

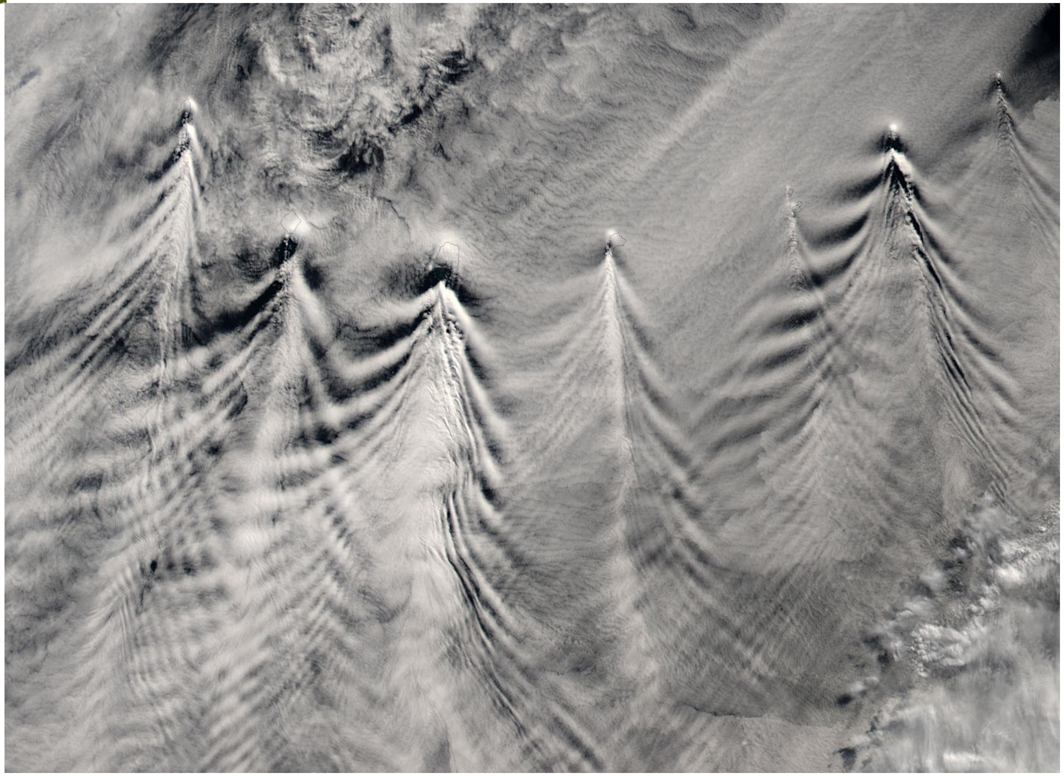
$$\boxed{\arctan\frac{1}{\sqrt{8}} \approx 19^\circ 28'}$$

Stationary phase, therefore prominent uncanceled waves, possible only between these angles.

This is the ***Kelvin wedge***, whose angle is $2 \times 19^\circ 28' \approx 39^\circ$.

Boat in a canal

Sandwich Islands, South Pacific :
clouds streaming past mountain peaks



Review of what we saw in lecture 2/3

- dispersion relations
- rings on water
- waves parallel to beach, surfing
- tsunami
- tide : out-of-phase response, upside-down pendulum
- Kelvin wedge : method of stationary phase