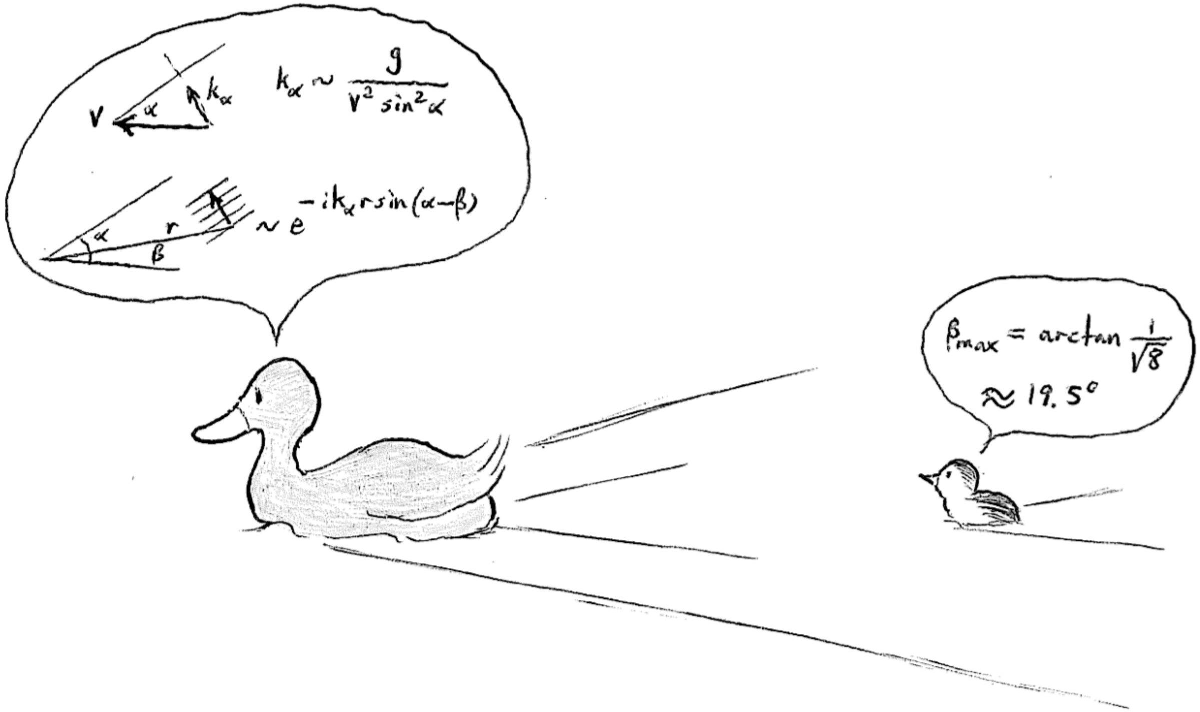
An invitation to simple modeling of complex phenomena



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Lyon, August 2012

A pure mathematician's healthy balance :

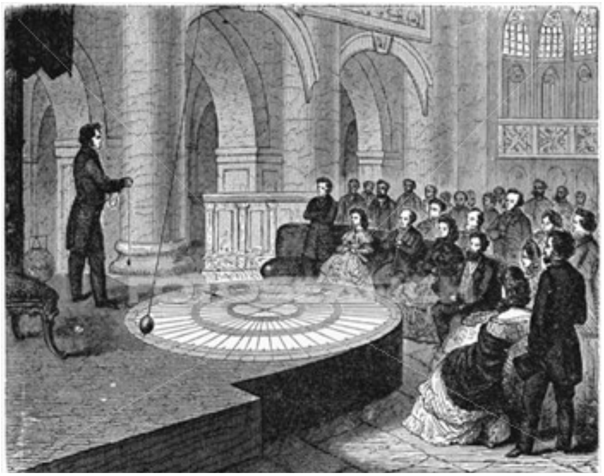
$$\#\text{examples} \gg \#\text{theorems} \gg \#\text{definitions}$$

An applied mathematician's healthy balance :

$$\#\text{phenomena explained/predicted} \gg \#\text{models} \gg \#\text{principles}$$

This balance is achievable only if we

- strive for simplicity as we go phenomena $\rightarrow$ models $\rightarrow$ principles
- forage for diversity as we go phenomena $\leftarrow$ models $\leftarrow$ principles.



An easy problem : a pendulum

What is the ***period*** ?

[Foucault’s pendulum, Panthéon, Paris]

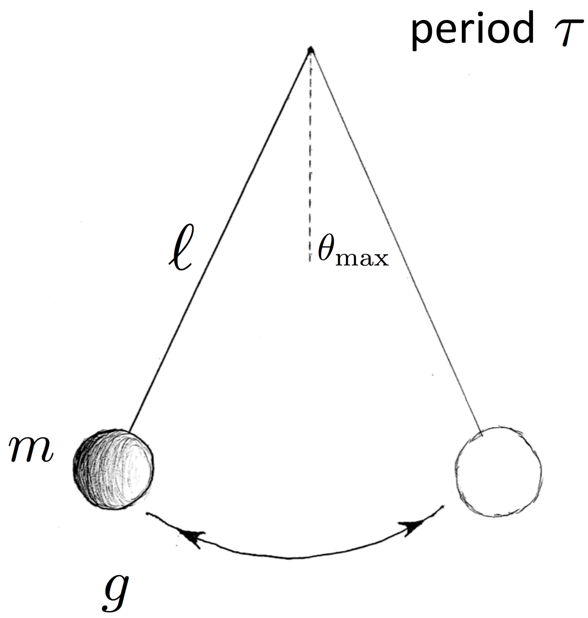
Three approaches to modeling :

1) *minimal: dimensional analysis*
(before we know anything)

2) *intermediate: back-of-the-envelope estimate*
(once we know something)

3) *maximal: solving the full equations*
(after we know everything already)

1) *Minimal: dimensional analysis*



variables	m	ℓ	g	$\theta_{\max}$	τ
dimensions (units)	$\mathbf{M}$	$\mathbf{L}$	$\dfrac{\mathbf{L}}{\mathbf{T}^2}$	$\mathbf{1}$	$\mathbf{T}$

[mass] = $\mathbf{M}$ [length] = $\mathbf{L}$ [time] = $\mathbf{T}$

5 variables, **3** basic dimensions

$\Longrightarrow$ 5 − 3 = **2** dimensionless groupings among variables

$$\Pi_1 = \frac{g\tau^2}{\ell} \sim 1 \qquad \Pi_2 = \theta_{\max} \sim 1$$

Underlying mathematical mechanism:

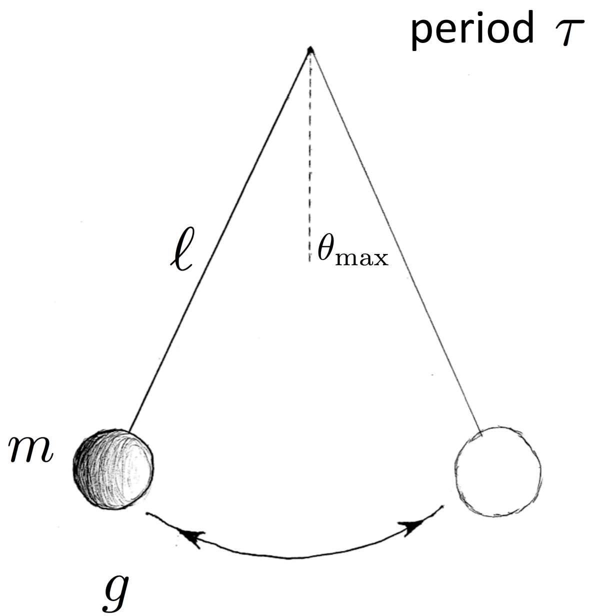
$$m^x \ell^y g^z {\theta_{\max}}^u \tau^v \sim 1$$

$$\mathbf{M}^x \mathbf{L}^{y+z} \mathbf{T}^{-2z+v} = \mathbf{M}^0 \mathbf{L}^0 \mathbf{T}^0$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

number of dimensionless groupings = $\boxed{\begin{array}{l}\text{nullity} \\ \text{dim} - \text{rank}\end{array}}$ = 5 − 3 = 2

i.e. just *linear algebra* .



$$\Pi_1 = \frac{g\tau^2}{\ell} \sim 1 \qquad \Pi_2 = \theta_{\max} \sim 1$$

Laws of nature must be expressible in ***dimensionless form*** :

$$F(\Pi_1, \Pi_2) = 0$$

or solving for Π_1 ,

$$\Pi_1 = f(\Pi_2)$$

$$\tau = \sqrt{\frac{\ell}{g} \cdot f(\theta_{\max})}$$

We did not have to think about physics, yet
we obtained the most interesting feature of the answer :

$$\boxed{\tau \sim \sqrt{\frac{\ell}{g}}}$$

For deeper dimensional analysis, see

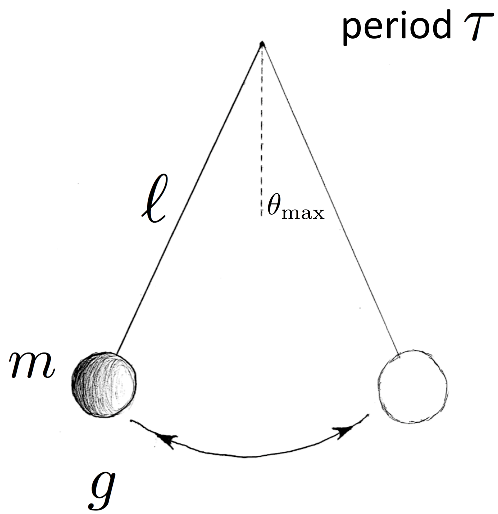
Barenblatt, *Scaling* (Cambridge UP)

It is surprisingly deep, leading to ***renormalization group*** ,
a powerful method in statistical physics, probability, etc.

Hierarchy of `equalities' :

$=$	exactly equal
$\approx$	size of coefficient near 1
$\sim$	*dimensionally correct* equality but no control over size of dimensionless coefficient
$\propto$	*proportional* but coefficient may have nontrivial dimension

2) *Intermediate: back-of-the-envelope estimate*



Assume $\theta_{\max} \ll 1$

Periodic oscillation near equilibrium

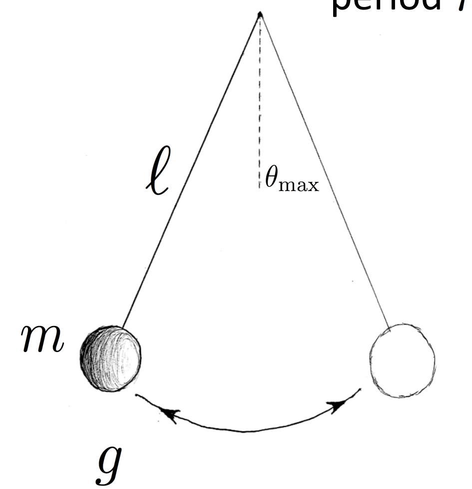
. . . model it as a ***spring*** !

$$\left.\begin{array}{ll} \text{restoring force} & \approx m\,g \\ \text{mass} & = m \\ \text{acceleration} & \approx \omega^2 \ell \end{array}\right\} \Longrightarrow \quad m\,\omega^2 \ell \approx m\,g$$

$$\Longrightarrow \quad \boxed{\tau = \frac{2\pi}{\omega} \approx 2\pi\sqrt{\frac{\ell}{g}}}$$

This time we thought a little about physics,
and identified the dimensionless coefficient 2π .

3) *Maximal: solving the full equation*



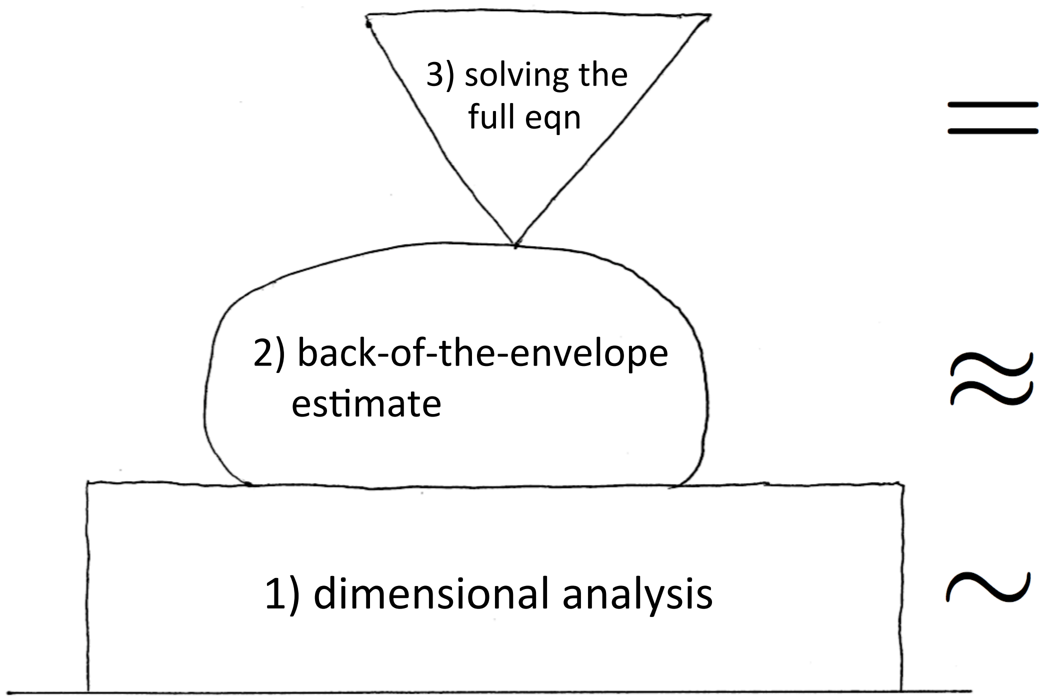
$$m\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\ell\,\theta = -m\,g\,\ell\,\sin\theta$$

$\dot{\theta}\times$ and integrate :

$$\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\ell}{g}}\frac{\mathrm{d}\theta}{\sqrt{k^2-\sin^2\frac{\theta}{2}}}\,, \qquad k=\sin\frac{\theta_{\max}}{2}$$

$$\Longrightarrow \quad \tau = 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,, \qquad \sin\frac{\theta}{2} = k\sin\varphi$$

$$= 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\mathrm{d}\varphi\Big(1+\frac{1}{2}k^2\sin^2\varphi+\cdots\Big) = \boxed{2\pi\sqrt{\frac{\ell}{g}}\Big(1+\frac{1}{4}k^2+\cdots\Big)}$$



complicated, precise
fragile, remote,
finished textbooks . . .

simple, approximate,
robust, close to phenomena,
research frontier. . .

In her/his research, a good applied mathematician should try

1) dimensional analysis on some new problem once a day

2) back-of-the-envelope estimate once a week

3) solving the full equations once a season

Thus, most of an applied mathematician's life is spent on doing 1) and 2), so it is urgent that you

get into the habit of

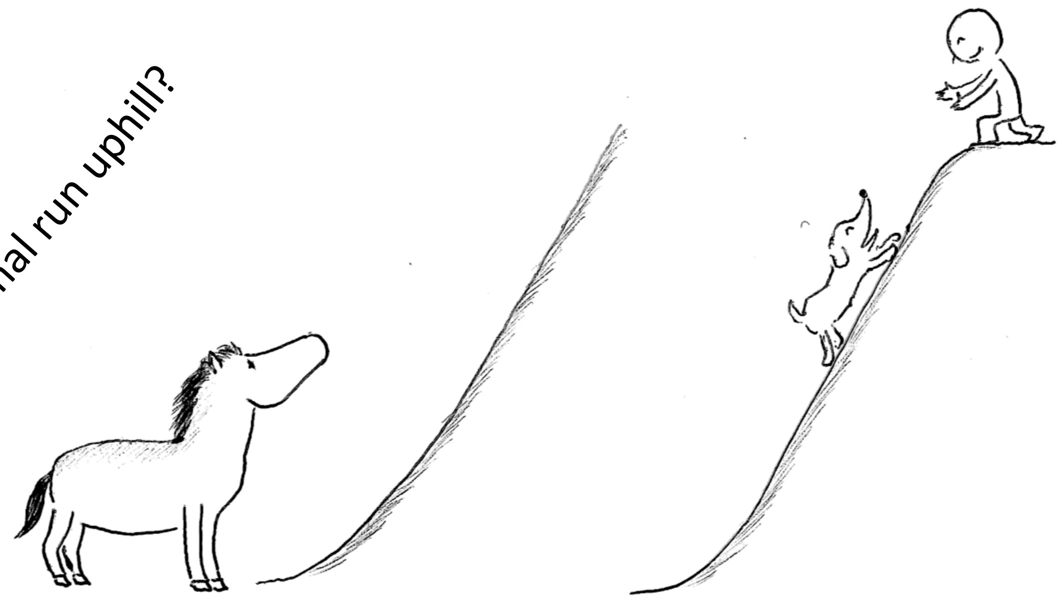
dimensional analysis +
back-of-the-envelope estimates

なかやすみ

Intermission

Biological applications

How fast can an animal run uphill?



How does the animal's ***power*** P depend on its size in length ℓ ?

- The cross-sectional area of its bones $\propto \ell^2$.
- The animal does not overheat spontaneously, but it radiates heat through its skin whose area $\propto \ell^2$.

For all these reasons,

$$P \propto \ell^2$$

When running uphill, the animal must lift its own ***weight*** $W \propto \ell^3$.

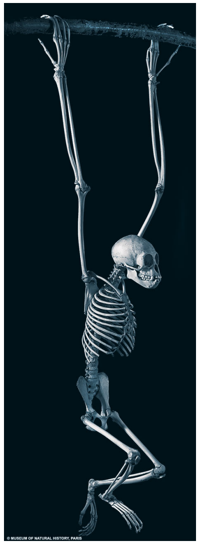
So the uphill speed $\propto \dfrac{P}{W} \propto \dfrac{\ell^2}{\ell^3} = \dfrac{1}{\ell}$.

This $\frac{P}{W} \propto \frac{1}{\ell}$ is a severe handicap for larger animals.

For example

- a mouse falling from a 2nd floor would feel nothing,
- a human might break a leg,
- an elephant would not survive.

Many compensate by having STURDIER skeletons than smaller animals.



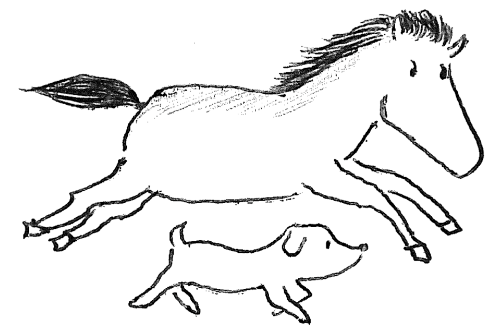
gibbon

gorilla

In contrast, when running on flat ground,
the ***drag*** (resistance force) D by the air varies like $\propto \ell^2$.

So the speed $\propto \dfrac{P}{D} \propto \dfrac{\ell^2}{\ell^2}$

is independent of ℓ

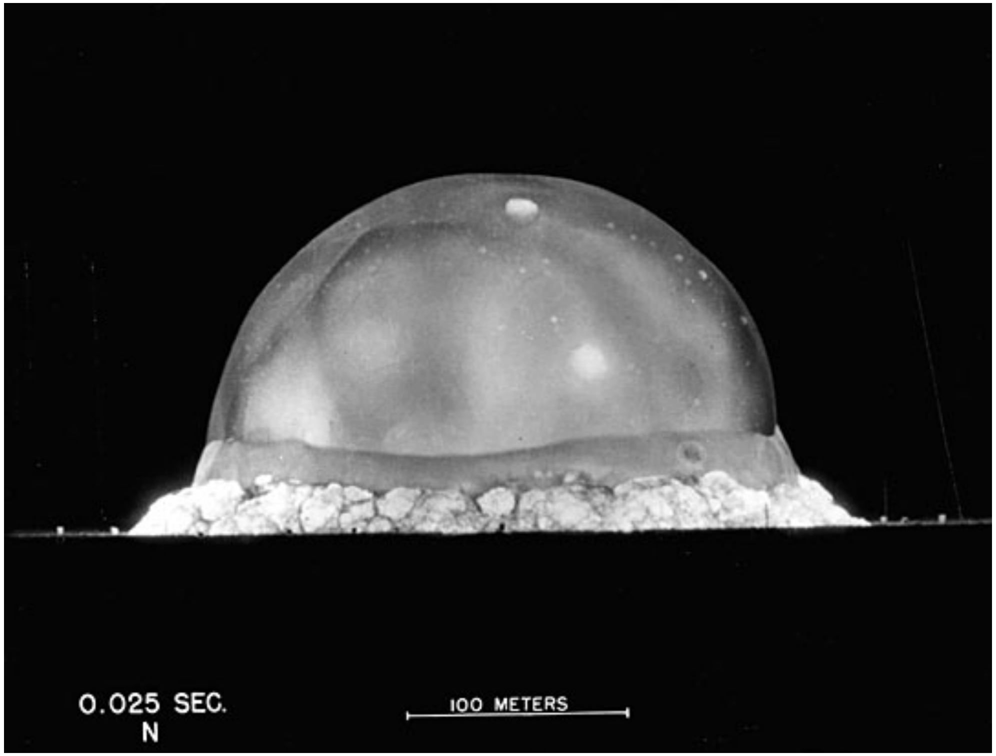


i.e. on flat ground all animals run at similar speeds.

Animal	Body mass, M (kg)	Leg length, L_0 (m)	Speed ($\mathrm{m\,s^{-1}}$)	Froude number $u/(\mathbf{g}L_0)^{0.5}$
Kangaroo rat	0.112	0.099	1.8	1.8
White rat	0.144	0.065	1.1	1.4
Tammar wallaby	6.86	0.33	3.0	1.7
Dog	23.6	0.50	2.8	1.3
Goat	25.1	0.48	2.8	1.3
Red kangaroo	46.1	0.58	3.8	1.6
Horse	135	0.75	2.9	1.1

(from C. Farley, et al., *J. exp. Biol.* 1993)

A *very* difficult problem



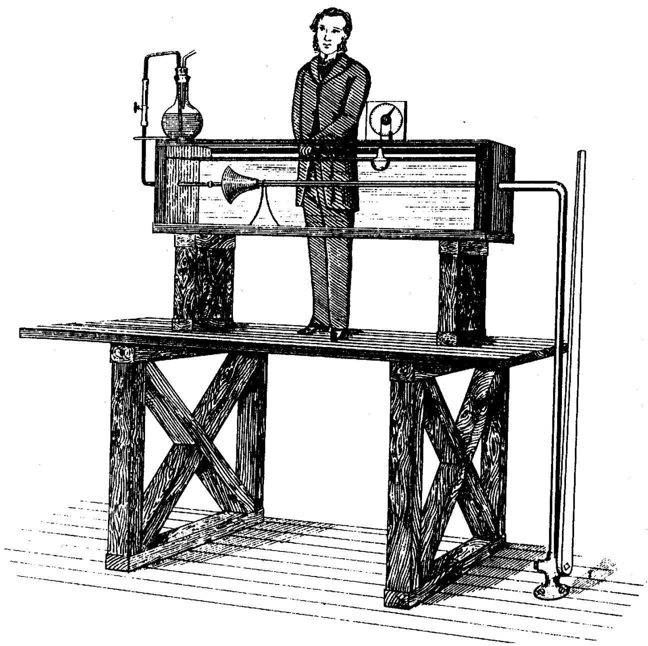
`Trinity test' – first man-made nuclear explosion
16 July 1945
desert of Jornada del Muerto, New Mexico.

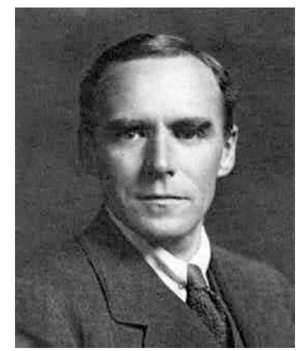
This photo appeared in newspapers, whereas
the ***energy*** of the explosion was classified top secret . . .

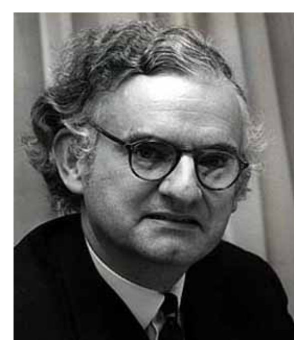
But G. I. Taylor estimated this energy
and published it in *Proc. Roy. Soc.* 1950, causing widespread embarrassment.

We shall now retrace G. I.'s argument,
a classic masterpiece of dimensional analysis.

G. I. Taylor (1886–1975):
this man

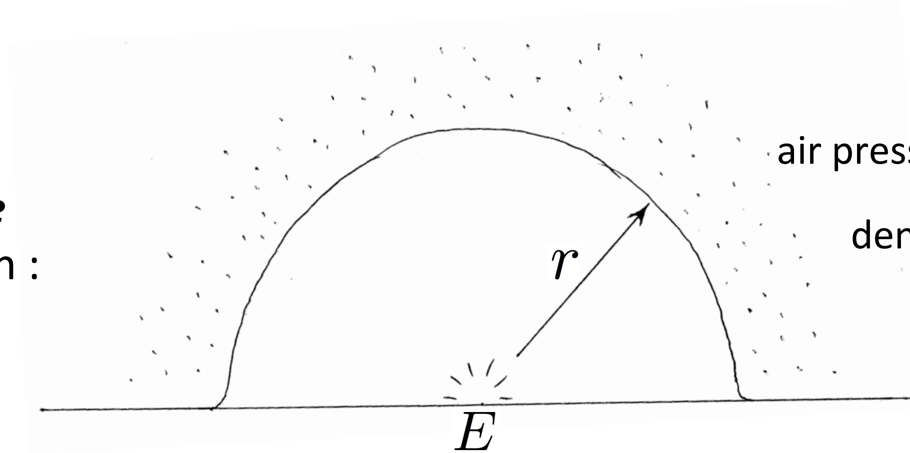






Some Cambridge fluid dynamicists . . .

Snapshot of the ***shock wave*** at time t after the explosion :



variables	t	r	$\cancel{p}$	ρ	E
dimensions	$\mathbf{T}$	$\mathbf{L}$	$\cancel{\dfrac{\mathbf{M}}{\mathbf{LT}^2}}$	$\dfrac{\mathbf{M}}{\mathbf{L}^3}$	$\dfrac{\mathbf{ML}^2}{\mathbf{T}^2}$

The shock wave is so intense that ρ matters, p does not (we think a little physics here).

4 variables, **3** basic dimensions,

$\Longrightarrow$ 4 – 3 = **1** dimensionless grouping $\Pi = \left(\dfrac{E\,t^2}{\rho}\right)^{\frac{1}{5}} \dfrac{1}{r}$

$$\Pi = \left(\frac{E\,t^2}{\rho}\right)^{\frac{1}{5}} \frac{1}{r}$$

$$F(\Pi) = 0$$

$$\Pi = \text{const.}$$

$$\boxed{E = (\text{const.}\, r)^5 \ \frac{\rho}{t^2}}$$

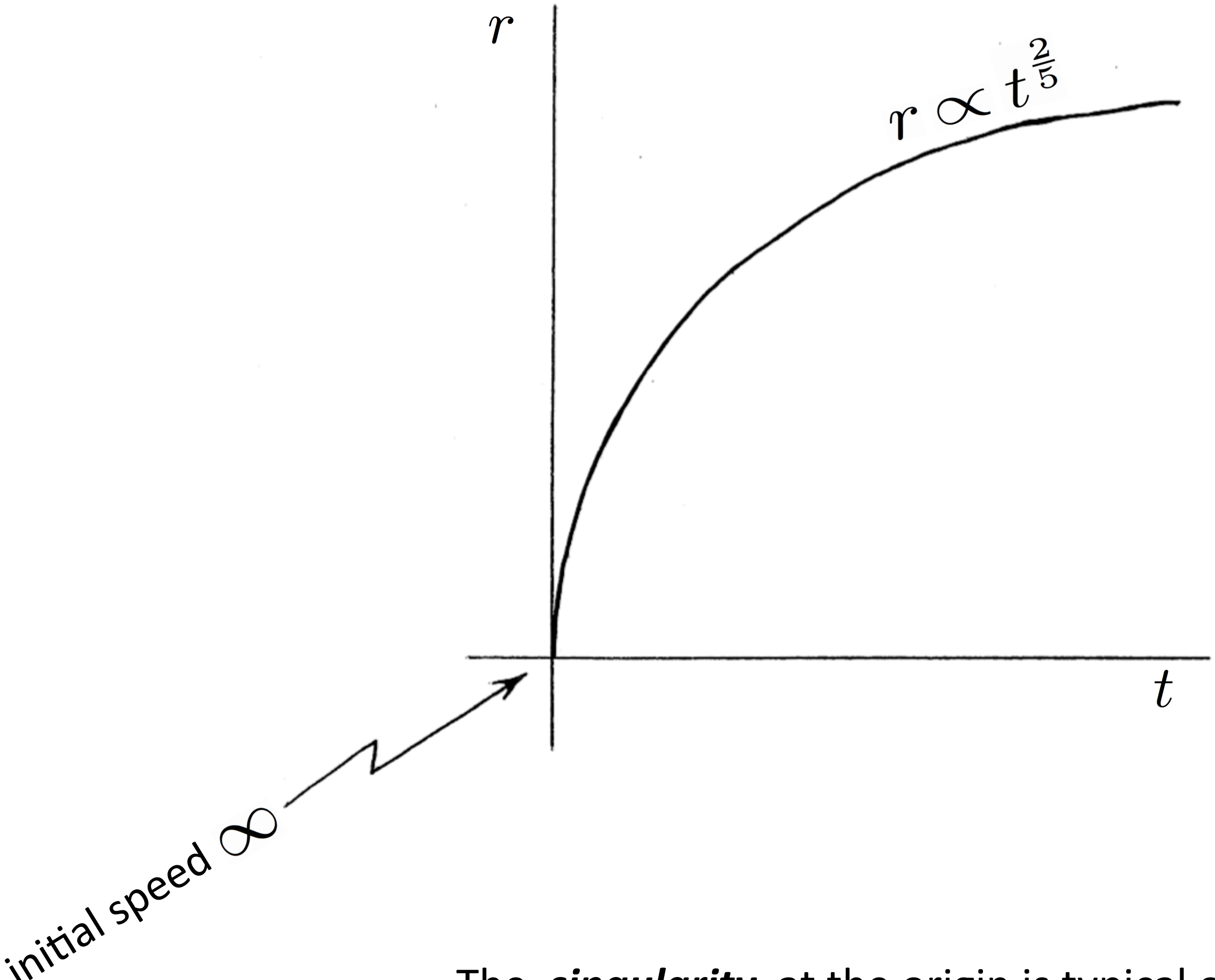
An experiment using a dynamite shows $\text{const.} \approx 1.$

Theoretically too we can show

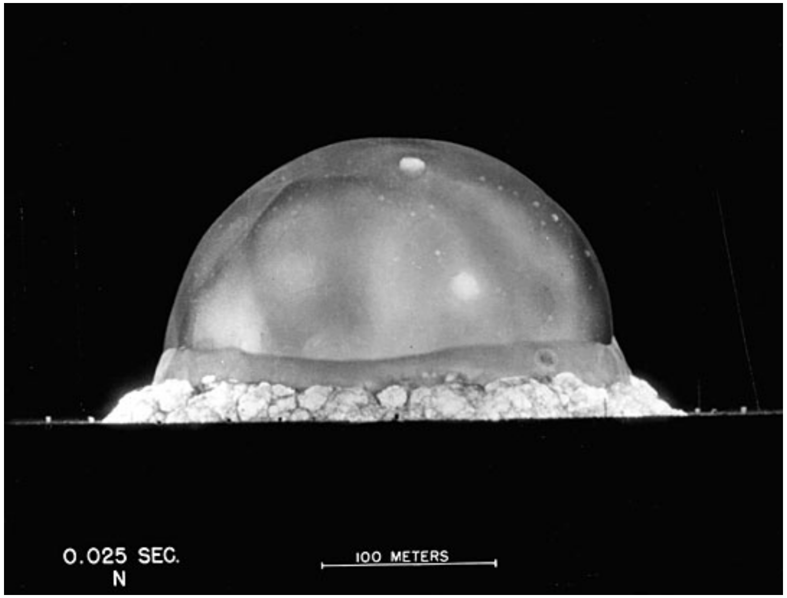
$$\text{const.} = \left(\frac{75(\gamma - 1)}{8\pi}\right)^{\frac{1}{5}} \approx 1.036$$

where $\gamma \approx 1.4$

For a fixed value of E in $E = (\mathrm{const.}\, r)^5 \, \dfrac{\rho}{t^2}$



The ***singularity*** at the origin is typical of explosive phenomena.
It came out automatically from dimensional analysis.



$$\boxed{E \approx \frac{r^5 \rho}{t^2}}$$

With

$$t = 2.5 \times 10^{-2}\,\mathrm{sec}$$
$$r = 1.4 \times 10^{2}\,\mathrm{m}$$
$$\rho = 1\,\mathrm{kg/m^3}\ \text{(at 1500 m altitude)}$$

we find

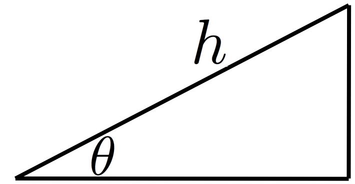
$$E \approx 86\,\mathrm{TJ}\ (\mathrm{terajoule},\, 10^{12}\,\mathrm{J})$$

Compare with the secret value, which was

$$84\,\mathrm{TJ}$$

$$\ldots\ \mathrm{error} < 2.5\,\%$$

A geometric application



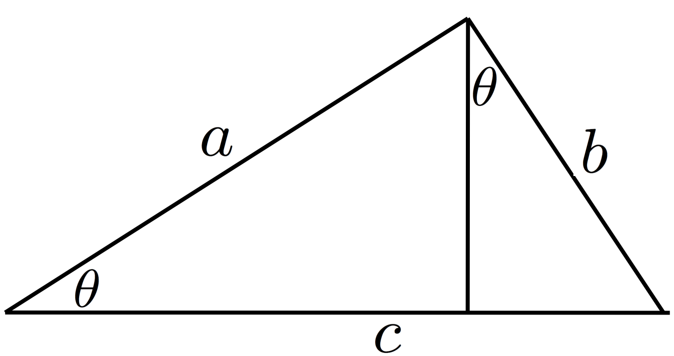
A right triangle is completely determined by its hypotenuse h and one of its acute angles θ.

In particular, its area A is determined.

The dimensionless groupings are $\Pi_1 = \dfrac{A}{h^2} \sim 1$ and $\Pi_2 = \theta \sim 1$.

$$\Pi_1 = f(\Pi_2)$$

$$A = h^2 f(\theta)$$ (in fact $f(\theta) = \dfrac{1}{2}\cos\theta\sin\theta$)

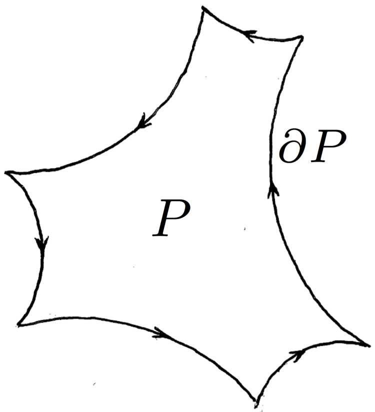


Here two triangles add to a large triangle

$$a^2 f(\theta) + b^2 f(\theta) = c^2 f(\theta)$$

Canceling $f(\theta)$, *we have Pythagoras*.

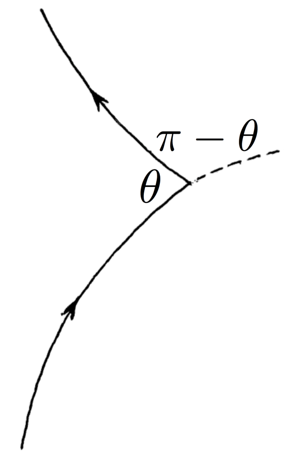
In spherical $K > 0$ or hyperbolic $K < 0$ geometry,
the Gauss-Bonnet formula



$$\iint_P K \, \mathrm{d}(\mathrm{area}) + \int_{\partial P} \kappa_g \, \mathrm{d}(\mathrm{length}) = 2\pi$$

implies that the area of a geodesic polygon
is determined by the sum of its angles :

$$K \cdot \mathrm{area}(P) \; + \; \sharp \, \mathrm{vertices}(P) \cdot \pi - \sum \theta = 2\pi$$



So P can never be decomposed into
smaller polygons similar to P .

In these non-Euclidean geometries,
no theorem of Pythagorean type that is *scaling-invariant* .

Review of what we saw in lecture 1/3

- importance of dimensional analysis and back-of-the-envelope estimates
- $\propto \quad \sim \quad \approx \quad =$
- period of pendulum
- animal running uphill, on flat ground
- nuclear explosion
- Pythagorean theorem