Viscous approximation : the Burgers equation
Resolution and convergence

1. The Burgers equation

- Friction and viscosity

Real fluids are never completely inviscid. They can be **weakly viscous**, with a so small viscosity that inviscid dynamics is expected to provide a good approximation.

Viscous dissipation is related to friction forces between the different slices of fluid : it can be expressed in terms of the stress tensor (cf for instance the Navier-Stokes equations).

Even negligible, the viscosity will play a role, especially for singular solutions, selecting **physically admissible singularities**.

Note that, for multidimensional system of conservation laws, this property is no more known to hold.

▸ Physical features of the Burgers equation

By analogy, we introduce a **viscous approximation** of the Hopf equation, referred to as Burgers equation :

$$\begin{aligned}\partial_t u + u\partial_x u = \frac{\varepsilon}{2}\partial^2_{xx} u,\\ u_{|t=0} = u_0\,,\end{aligned} \tag{1}$$

where ε is a small parameter.

Multiplying this equation by u and integrating with respect to t and x, we get

$$\frac{1}{2}\int u^2(t,x)dx + \frac{\varepsilon}{2}\int_0^t \int (\partial_x u)^2(s,x)dxds = \frac{1}{2}\int u_0^2(x)dx\,,$$

provided that u decays sufficiently at infinity.
The entropy (also called energy in this unphysical context) is dissipated and the evolution is irreversible. We further have an **explicit formula for the entropy dissipation**.

For fixed ε, we expect that

- the **regularizing effect** of the heat equation should master the nonlinearity ;
- shocks should be smooth in **viscous profiles** of thickness $\sqrt{\varepsilon}$.

We will actually prove the existence and uniqueness of a global solution u_ε for any continuous and bounded initial data u_0. The maximum principle will further provide a uniform bound on u_ε.

As ε tends to 0, we expect that

- the sequence (u_ε) should converge (up to extraction) in some weak sense ;
- any limit point should satisfy the Hopf equation.

We will establish this convergence and will further show that this approximation **selects a unique weak solution** of the Hopf equation.

On this basic example, the proofs are relatively elementary since the solutions to the Burgers equation can be computed almost explicitly.

▸ The Hopf-Cole transformation

Let ϕ be some (classical) nonnegative solution of the **heat equation**

$$\partial_t \phi - \frac{\varepsilon}{2}\partial^2_{xx}\phi = 0.$$

Define U by $\phi(t,x) = \exp(-\lambda U(t,x))$. Then

$$\partial_t \phi - \frac{\varepsilon}{2}\partial^2_{xx}\phi = -\lambda \exp(-\lambda U)\left(\partial_t U + \frac{1}{2}(\varepsilon\lambda)(\partial_x U)^2 - \frac{\varepsilon}{2}\partial^2_{xx} U\right) = 0$$

Define $u = \partial_x U$. Then

$$\partial_t u + \frac{1}{2}(\varepsilon\lambda)\partial_x(u^2) - \frac{\varepsilon}{2}\partial^2_{xx} u = 0$$

If $\varepsilon\lambda = 1$, u is a solution to the **Burgers equation**.

Our strategy to solve the Burgers equation is therefore

- to **compute the initial data** ϕ_0

$$\phi_0 = \exp(-\frac{1}{\varepsilon} U_0) \text{ with } U_0(x) = \int_{-\infty}^{x} u_0(y)dy$$

  which is well defined if u_0 is smooth and rapidly decaying ;

- to **solve the heat equation** with initial data ϕ_0

- to retrieve the solution of the Burgers equation by **taking some logarithmic derivative** of ϕ :

$$u = -\varepsilon \frac{\partial_x \phi}{\phi}.$$

2. The heat equation

- Convolution and heat kernel

Since the heat equation is linear, all solutions can be obtained from the **Green function** G, i.e. from the fundamental solution of the heat equation having the Dirac mass as initial data

$$\begin{aligned} \partial_t G - \frac{\varepsilon}{2}\partial_{xx} G = 0, \\ G_{|t=0} = \delta\,. \end{aligned} \tag{2}$$

It is indeed easy to check that the function ϕ **defined by convolution**

$$\phi(t,x) = \int G(t,x-y)\phi_0(y)dy$$

satisfies the heat equation, and the initial condition $\phi(0,x) = \phi_0(x)$. (The formulation is not completely correct since $G_{|t=0}$ is not a function, but it can be made rigorous using the convolution of distributions.)

▸ Self-similar solutions

We will seek G as a self-similar solution of the form

$$G(t,x) = \frac{1}{\sqrt{\varepsilon t}} g\left(\frac{x}{\sqrt{\varepsilon t}}\right).$$

This choice is indeed consistent with

- the **scaling invariance** of the heat equation
  Let $u \equiv u(t,x)$ be a solution of the heat equation

  $$\partial_t u - \frac{\varepsilon}{2}\partial^2_{xx} u = 0\,, \quad u_{|t=0} = u_0\,.$$

  Then, for any $\lambda \in \mathbb{R}$, $u_\lambda \equiv u(\lambda^2 t, \lambda x)$ is also a solution with initial data $u_\lambda(0,x) = u_0(\lambda x)$.

- the **particular form of the initial data**

  $$\frac{1}{\sqrt{\varepsilon t}} g\left(\frac{x}{\sqrt{\varepsilon t}}\right) \to \delta \text{ in the sense of distributions as } t \to 0$$

  for any fixed ε and any probability density g.

Easy computations lead then to the **ordinary differential equation**

$$-\frac{\varepsilon}{2}\frac{1}{(\varepsilon t)^{3/2}}g\left(\frac{x}{\sqrt{\varepsilon t}}\right)-\frac{\varepsilon}{2}\frac{x}{(\varepsilon t)^{5/2}}g'\left(\frac{x}{\sqrt{\varepsilon t}}\right)-\frac{\varepsilon}{2}\frac{1}{(\varepsilon t)^{3/2}}g''\left(\frac{x}{\sqrt{\varepsilon t}}\right)=0$$

which can be rewritten

$$g(\xi)+\xi g'(\xi)+g''(\xi)=0\,.$$

One can check that **Gaussian distributions** satisfy that ODE. The profile g has further to be normalized.

The **Green kernel** is therefore given by

$$G(t,x)=\frac{1}{\sqrt{2\pi\varepsilon t}}\exp\left(-\frac{x^2}{2\varepsilon t}\right)\,.$$

▸ Maximum principle and regularity

From the formula

$$\phi(t,x) = \int \frac{1}{\sqrt{2\pi\varepsilon t}} \exp\left(-\frac{(x-y)^2}{2\varepsilon t}\right) \phi_0(y)dy \,,$$

we deduce that ϕ is bounded by $\|\phi_0\|_\infty$, and that $\phi(t) \in C^\infty(\mathbb{R})$ for all $t > 0$. Indeed the integrand can be derived infinitely many times, and Lebesgue's theorem shows that

$$\partial_x^k \phi(t,x) = \int \frac{1}{\sqrt{2\pi\varepsilon t}} \partial_x^k \left(\exp\left(-\frac{(x-y)^2}{2\varepsilon t}\right)\right) \phi_0(y)dy \,.$$

The regularity of the solution does not depend on the regularity of the initial data. This **smoothing effect** is typical from parabolic PDEs.

- Uniqueness

As the **heat equation is linear**, it is enough to prove that zero is the only solution with zero initial data.

For the sake of simplicity, we will consider only solutions which decay at infinity (but uniqueness still holds in a larger class of solutions). Furthermore, without loss of generality, we can assume that these solutions are smooth.

Multiplying by ϕ the equation

$$\partial_t \phi - \frac{\varepsilon}{2} \partial^2_{xx} \phi = 0$$

and integrating with respect to t and x leads to the **energy equality**

$$\frac{1}{2} \int \phi^2(t,x)dx + \frac{\varepsilon}{2} \iint (\partial_x \phi)^2(s,x) dsdx = \frac{1}{2} \int \phi_0^2(x)dx = 0$$

from which we deduce that ϕ is identically zero.

3. The inviscid limit

Using the results on the heat equation together with the Hopf-Cole transformation, we get the existence of a unique solution u_ε to the Burgers equation (1) for any fixed $\varepsilon > 0$, and any nice initial data u_0. The next step is to describe the asymptotic behavior of u_ε as $\varepsilon \to 0$.

Proposition. *In the limit $\varepsilon \to 0$, (u_ε) converges (in a weak sense) to a function u, bounded almost everywhere on $\mathbb{R}^+ \times \mathbb{R}$. Furthermore,*

- *Discontinuity points of u are countable ;*
- *u is a global solution to the Hopf equation in the sense of distributions ;*
- *u satisfies the Lax-Oleinik condition (which guarantees uniqueness)*

$$\partial_x u(t,.) \leq \frac{1}{t}\,.$$

▸ A stationary phase argument

The proof is based on the **explicit formula** for u_ε :

$$u_\varepsilon(t,x) = \int u_0(y) d\mu_\varepsilon(t,x,y)$$

where the probability measure μ_ε is defined by

$$d\mu_\varepsilon(t,x,y) = \frac{\exp\left(-\frac{1}{\varepsilon}U_0(y) - \left(\frac{x-y}{\sqrt{2\varepsilon t}}\right)^2\right) dy}{\int \exp\left(-\frac{1}{\varepsilon}U_0(y) - \left(\frac{x-y}{\sqrt{2\varepsilon t}}\right)^2\right) dy}.$$

In the limit $\varepsilon \to 0$, the measure μ_ε shall concentrate on **minimal points of the phase**

$$\Psi(t,x,y) = U_0(y) + \frac{1}{2t}(x-y)^2 .$$

▸ Discontinuity points

As $\Psi(t,x,\cdot)$ is smooth and tends to infinity at $\pm\infty$, the minimum of $\Psi(t,x,\cdot)$ is attained on a **compact set**

$$I(t,x) \subset \{y \in \mathbb{R} \,/\, u_0(y) - \frac{1}{t}(x-y) = 0\}\,.$$

so that we can define $y_-(t,x) = \min I(t,x)$ and $y_+(t,x) = \max I(t,x)\,.$

What can be proved is the following elementary property

$$\forall x_1 < x_2, \quad y_+(t,x_2) \geq y_+(t,x_2) \geq y_+(t,x_1) \geq y_+(t,x_1)$$

so that, for any $t > 0$, the functions $y_+(t,\cdot)$ and $y_-(t,\cdot)$ are non decreasing and **coincide outside from a countable set** S_t.

Outside from S_t, $\mu_\varepsilon(t,x,\cdot)$ therefore concentrates on a **Dirac mass** at $y_-(t,x) = y_+(t,x)$.

▸ Stability of the integral equation

In other words, for any $t > 0$, $(u_\varepsilon(t,.))$ **converges on** $\mathbb{R} \setminus S_t$ towards the function $u(t,.)$ defined by

$$u(t,x) = u_0(y_\pm(t,x)) = \frac{1}{t}(x - y_\pm(t,x))\,.$$

Starting from the **weak formulation** of the Burgers equation

$$\int_0^{+\infty} \int \left(u_\varepsilon \partial_t \varphi + \frac{1}{2} u_\varepsilon^2 \partial_x \varphi + \frac{\varepsilon}{2} u_\varepsilon \partial_{xx} \varphi \right) dtdx = - \int u_0 \varphi_{|t=0} dx\,,$$

and taking limits as $\varepsilon \to 0$ leads then to

$$\int_0^{+\infty} \int \left(u \partial_t \varphi + \frac{1}{2} u^2 \partial_x \varphi \right) dtdx = - \int u_0 \varphi_{|t=0} dx$$

which is the weak formulation of the Hopf equation.

▸ Lax-Oleinik's condition

The function $x \mapsto y_+(t,x)$ is continuous outside from the countable set S_t, and locally bounded. It can be therefore considered as a distribution, and has weak derivatives. As $x \mapsto y_+(t,x)$ **is non decreasing**,

$$\partial_x y_+ \geq 0 \text{ in the sense of distributions.}$$

From the identity defining the **minimal points of the phase**

$$u_0(y_+(t,x)) - \frac{1}{t}(x - y_+(t,x)) = 0$$

we then deduce the Lax-Oleinik condition

$$t\partial_x u = 1 - \partial_x y_+ \leq 1 \text{ in the sense of distributions.}$$

Proving that this condition guarantees **uniqueness** is much more difficult and is beyond the scope of this lecture.