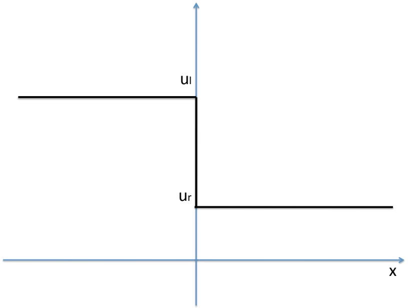
The Riemann problem
Rarefaction waves and shock waves

1. An illuminating example

▸ A Heaviside function as initial datum

Solving the Riemann problem for the Hopf equation consists in describing the solutions to

$$\begin{cases} \partial_t u + \frac{1}{2}\partial_x u^2 = 0, \\ u_0(x) = u_l \text{ for } x \leq 0, \quad u_0(x) = u_r \text{ for } x > 0\,. \end{cases} \tag{1}$$



- Scaling invariance

Let $u \equiv u(t,x)$ be a solution of the (conservative) Hopf equation

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0\,,$$

with initial data u_0. Then, for any $\lambda \in \mathbb{R}$, $u_\lambda \equiv u(\lambda t, \lambda x)$ is also a solution with initial data

$$u_\lambda(0,x) = u_0(\lambda x)\,.$$

Note that, because of the homogeneity of the flux function, the Hopf equation admits other scaling invariances. But this is not a general feature of hyperbolic system of conservation laws.

- Self-similar solutions

Because of the scaling invariance of the equation and the particular form of the initial data, it is then natural to seek **self-similar solutions** of the form

$$u(t,x) = v\left(\frac{x}{t}\right) .$$

The initial data prescribes the **limiting values**

$$\lim_{z \to -\infty} v(z) = u_l,$$
$$\lim_{z \to +\infty} v(z) = u_r .$$

The partial differential equation reduces to an **ordinary differential equation** (possibly in weak form to catch singularities).

Glimm's approximation scheme

Solving the Riemann problem is not only an exercise. It is the elementary step to prove the **global existence of weak solutions** for 1D hyperbolic system of conservation laws.

A discrete scheme has been indeed proposed by Glimm to get **approximate solutions**.
For each time step $nh \to (n+1)h$

- define spatial cells of size k
- choose (randomly) one point in each half interval
- compute the values of u at time nh at these two points
- solve Riemann's problem on each cell

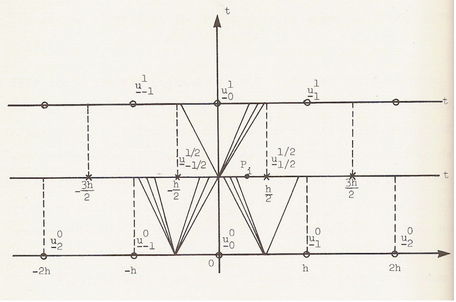


Fig. 2. Glimm's scheme

The convergence of this approximation scheme as $h, k \to 0$ then requires to obtain

- **a priori bounds** (using the explicit wave profiles) ;
- **compactness** (coming from an additional bound on the interaction potential and the total variation).

Such a proof is beyond the scope of this lecture. For scalar equations, we will see (in the next parallel session) an alternative proof of existence based on some viscous approximation.

2. "Smooth" solutions : rarefaction waves

▸ A simple ordinary differential equation

For "smooth" solutions, (1) can be rewritten

$$\begin{cases} -\dfrac{x}{t^2}v'\left(\dfrac{x}{t}\right) + \dfrac{1}{t}v\left(\dfrac{x}{t}\right)v'\left(\dfrac{x}{t}\right) = 0, \\ \lim\limits_{z\to-\infty} v(z) = u_l, \quad \lim\limits_{z\to+\infty} v(z) = u_r\,. \end{cases} \tag{2}$$

For continuous, piecewise C^1 functions, the ordinary differential equation leads to the condition

$$v'(z) = 0 \text{ or } v(z) = z\,.$$

The **wave profile** is then given by a combination of affine functions.

Note that the equation is not really satisfied in strong sense, insofar as the identity holds only **almost everywhere**.

We finally get a **unique solution**

$$v(z) = z \text{ if } z \in [u_l, u_r], \quad v(z) = u_l \text{ if } z \le u_l \,,$$
$$\text{and } v(z) = u_r \text{ if } z \ge u_r \,.$$

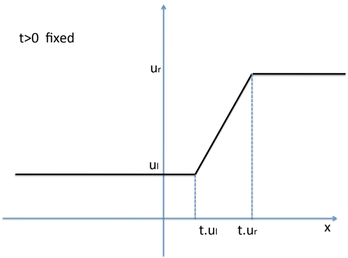


Fig. 3. Rarefaction wave

The rarefaction fans connect only **states** u_l **and** u_r **such that**

$$u_l \le u_r \,.$$

▸ Characteristics

The method of characteristics provides

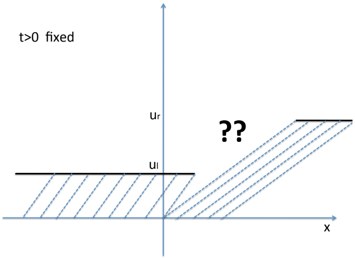


Fig. 4. Characteristics for the Riemann problem

The problem is underdetermined. We will see actually that there exist other weak solutions.

▸ A singularity at time 0

A natural way to remove the uncertainty is to regularize the initial profile. Provided that the regularized data $u_{0,\varepsilon}$ is monotonic increasing, there exists a global smooth solution. According to the previous lecture, we indeed have

$$t_{*,\varepsilon} = \frac{1}{\max_{x\in\mathbb{R}}(-u'_{0,\varepsilon}(x))_+} = +\infty\,.$$

It is therefore relevant to get an initial **singularity which does not propagate**, since - in a generalized sense -

$$\frac{1}{\max_{x\in\mathbb{R}}(-u'_0(x))_+} = +\infty\,.$$

- Conservation of entropies

Since the rarefaction profile v is a continuous, piecewise C^1 solution of the Riemann problem (2), it is possible to multiply the ODE by $F'(v)$ for any smooth F to get

$$-\frac{x}{t^2}F'\left(v\left(\frac{x}{t}\right)\right)v'\left(\frac{x}{t}\right)+\frac{1}{t}v\left(\frac{x}{t}\right)F'\left(v\left(\frac{x}{t}\right)\right)v'\left(\frac{x}{t}\right)=0\,,$$

In other words, denoting by G any function such that $F'(z)z = G'(z)$,

$$\partial_t F\left(v\left(\frac{x}{t}\right)\right)+\partial_x G\left(v\left(\frac{x}{t}\right)\right)=0\,.$$

The entropies are conserved, **the evolution is reversible** (as long as the initial time is not reached).

3. Discontinuous solutions : shock waves

- ## The Rankine-Hugoniot jump conditions

In the case when $u_l > u_r$, there is no continuous self-similar solution to the Riemann problem. A natural idea is then to weaken the differential condition, and to require that only an integral version of the conservation law is satisfied.

More precisely, we will search a solution in the form of a **Heaviside function** (like the initial data)

$$v(z) = u_l \text{ if } z \leq s, \quad v(z) = u_r \text{ if } z > s\,.$$

Simple computations then lead to the following jump condition

$$-s(u_r - u_l) + \frac{1}{2}(u_r^2 - u_l^2) = 0$$

referred to as **Rankine-Hugoniot condition**.

The **shock speed** s does not coincide with the velocity of particles : there is a kind of rearrangement process in order that the wave front is stable.

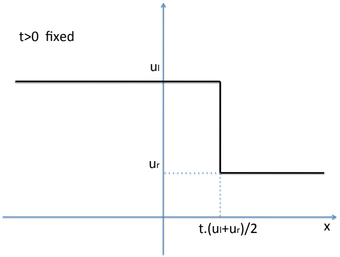


Fig. 5. Shock wave

The sound barrier is a well-known example of such a shock wave. It corresponds to the point at which an aircraft moves from transonic to supersonic speed.

▸ Characteristics

The method of characteristics provides

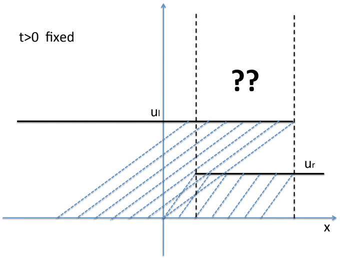


Fig. 5. Characteristics for the Riemann problem

The problem is overdetermined. We need a kind of averaging process to define weak solutions.

- Lax-Oleinik's condition

Even if we regularize the initial profile, a singularity does appear for finite time

$$t_{*,\varepsilon} = \frac{1}{\max_{x\in\mathbb{R}}(-u'_{0,\varepsilon}(x))_+} \to 0 \text{ as } \varepsilon \to 0\,.$$

In other words, singularities corresponding to decreasing discontinuities are stable.

We will actually see (in the next parallel session) that a natural condition for singular solutions to be admissible is the **Lax-Oleinik criterion**

$$t\partial_x u \leq 1\,,$$

which is inherited from the **microscopic structure of the singularities**.

- Decay of entropies

Let $F(z) = \frac{1}{2}z^2$ and $G(z) = \frac{1}{3}z^3$ be some associated flux, i.e. a function such that $F'(z)z = G'(z)$. Straightforward computations lead to

$$\begin{aligned} & -s(F(u_r) - F(u_l)) + (G(u_r) - G(u_l)) \\ & \quad = -\frac{1}{2}(u_r + u_l)(F(u_r) - F(u_l)) + (G(u_r) - G(u_l)) \\ & \quad = (u_r - u_l)(-\frac{1}{4}(u_r^2 + u_l^2 + 2u_r u_l) + \frac{1}{3}(u_r^2 + u_r u_l + u_l^2)) \\ & \quad = (u_r - u_l)(\frac{1}{12}(u_r^2 + u_l^2) - \frac{1}{6}u_r u_l) < 0 \end{aligned}$$

Note that a similar argument holds for any convex entropy.

The mathematical entropy (which is the opposite of the physical entropy) is therefore a decreasing function of time. **The evolution is irreversible**.