Solutions for some of the exercises of Ken Ono (1)

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If you find any mistakes, please email me. Also if after reading this you still can't do the exercises, email me too, and I will provide complete answers. This page will eventually contain a sketch of the answers to exercise 3.

Recall the definition of the partition function :

$$p(n) = \#\{k_1 \geq \cdots \geq k_l \geq 1 : n = k_1 + \cdots + k_l\}.$$

So for instance $p(4) = 4$, because $4 = 4 = 3+1 = 2+1+1 = 1+1+1+1$. We now need to find the **generating function** of $p(n)$, which is by definition the following power series :

$$f(q) = \sum_{n=0}^{+\infty} p(n) q^n$$

Why would we be interested in such functions ? Because they often have nicer expressions, which happen to be very handy. For $p(n)$, we actually have the following formula :

$$\sum_{n=0}^{+\infty} p(n) q^n = \prod_{n=1}^{+\infty} \frac{1}{1-q^n}.$$

In order to prove this, we need the following : $\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$. Now we can rewrite $\prod_{n=1}^{+\infty} \frac{1}{1-q^n}$ as

$$\prod_{n=1}^{+\infty} \frac{1}{1-q^n} = (1+q+q^2+\cdots)(1+q^2+q^4+\cdots)(1+q^3+q^6+\cdots)\cdots.$$

So to find the coefficients of the power series development of this product, we have to count how many times each q^n appears as ta product. We want to show this number is $p(n)$. The key is to rewrite a partition of n as

$$n = \underbrace{1+\cdots+1}_{m_1 \text{ times}} + \underbrace{2+\cdots+2}_{m_2 \text{ times}} + \cdots + \underbrace{k+\cdots+k}_{m_k \text{ times}} = m_1 \cdot 1 + \cdots + m_k \cdot k,$$

we associate to it the product of q^{m_1} from the first sum, q^{2m_2} from the second sum,..., q^{km_k} from the k'th sum, to get indeed $q^{m_1+2m_2+\cdots+km_k} = q^n$. Check that this actually lists all the ways of getting some q^n.

Exercise 1. Prove that there are infinitely many n's such that $p(n)$ is even, and infinitely many n's such that $p(n)$ is odd. You may use Euler's recurrence (without proving it!) :

$$p(n) + \sum_{k=1}^{\infty} (-1)^k \left[p(n - \frac{k(3k+1)}{2}) + p(n - \frac{k(3k-1)}{2}) \right] = 0.$$

Remark that the infinite sum over the k's is actually finite, as $p(m) = 0$ for $m \leq 0$.

Sketch of the answer. This is a proof by contradiction, in the same spirit as Euclid's proof that there are infinitely many prime numbers. So suppose for instance that there are only finitely many n's such that $p(n)$ is even. Let N be the greatest number for which $p(N)$ is even. The key thing is that the "gaps" between *all* the $k(3k \pm 1)/2$ goes to infinity when k tends to infinity, in particular there is k_0 such that the gap is greater than $N+1$ for $k \geq k_0$. Now for $n >> N$ well chosen ($n - \frac{k_0(3k_0-1)}{2} = N+1$ for instance), $n - \frac{k(3k\pm1)}{2}$ will never be in between 0 and N, by the above formula $p(n)$ is a sum of terms which are either (odd+odd) or 0, hence even. But then $p(n)$ is even, a contradiction as $n > N$.

And if there are finitely many n's such that $p(n)$ is odd, do almost the same trick by this time finding n such that there will be a unique $k \geq k_0$ and such that $n - k(3k+1)/2$ is in between 0 and N, and such that for this k, $n - k(3k+1)/2 = 1$. Then show that for some $n > N$ $p(n)$ has the parity of $p(1) = 1$, a contradiction.

Exercise 2. Show the following formula, which will be useful for exercise 3 :

$$\prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{(k^2+k)/2}.$$

You may use Jacobi's triple product formula (still without proving it!) :

$$\prod_{n=1}^{\infty} (1-x^{2n})(1-x^{2n-1}z)(1-x^{2n-1}z^{-1}) = \sum_{n=-\infty}^{+\infty} x^{n^2} z^n.$$

Sketch of the answer. The first thing you want to do to get some q^n's is to make the following change of variables : $x = \sqrt{q}$ and $z = -\sqrt{q}$, so that the expression on the left of Jacobi's triple product formula becomes

$$\prod_{n=1}^{+\infty} (1-q^n)(1-q^n)(1-q^{n-1}).$$

Unfortunately as you can see when $n = 1$, we have the last term $(1 - q^0) = 0$, so the whole product is null, and we are proving $0 = 0$, which is not really deep. So we want to somehow remove that term, and one way to do this is to still put $x = \sqrt{q}$, but let z vary for a while. We first get

$$\prod_{n=1}^{+\infty} (1-q^n)(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1}) = \sum_{n=-\infty}^{+\infty} q^{n^2/2} z^n.$$

Now we want to put the term which would cancel the product if $z = -\sqrt{q}$ on the right side, so we divide everything by $(1 + q^{1/2}z^{-1})$, which is $(1 + q^{n-1/2}z^{-1})$ for $n = 1$. Now the left term is

$$\prod_{n=1}^{+\infty} (1-q^n)(1+q^{n-1/2}z)^2,$$

which tends to the promised $\prod_{n=1}^{+\infty}(1-q^n)^3$ when $z \to -\sqrt{q}$. Now use L'Hospital's lemma. This says if f and g are derivable functions such that $f(x) \to 0$ and $g(x) \to 0$ when $x \to a$, then

$$\frac{f(x)}{g(x)} \to \frac{f'(a)}{g'(a)},$$

whenever this last fraction makes sense.