Islamic Art, Architecture, Mathematics, and Chemistry.

Yves Meyer

LYON

le 27 Août 2012

Dostoyevsky once let drop the enigmatic phrase :

“Beauty will save the world."

What does this mean ? For a long time it used to seem to me that this was a mere phrase. Just how could such a thing be possible ? When had it ever happened in the bloodthirsty course of history that beauty had saved anyone from anything ? Beauty had provided embellishment certainly, given uplift–but whom had it ever saved ?

Aleksandr Solzhenitsyn, Nobel Prize in literature, 1970.

(1) A Nobel prize laureate
(2) A few pictures of tilings
(3) Periodic tilings of the plane
(4) The pinwheel tiling
(5) Local patches and repetitive tilings
(6) Finitely generated tilings
(7) Delone sets
(8) Almost lattices, model sets, and quasicrystals
(9) Pisot-Vijayaraghavan numbers and Salem numbers
(10) Diffraction images of quasicrystals

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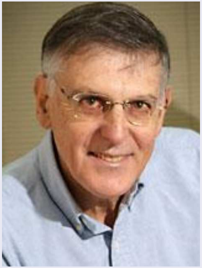
A Nobel prize laureate

The Nobel prize in chemistry 2011

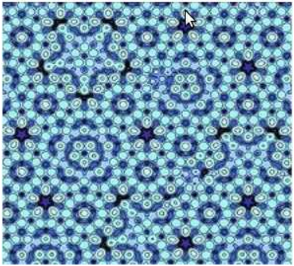
Dan Shechtman Technion - Israel Institute of Technology, Haifa, Israel

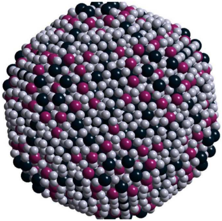
"for the discovery of quasicrystals"



Scientists believed that crystals in materials are always periodic.

A periodic crystal with fivefold ($2\pi/5$) symmetry is impossible. On that morning in 1982, Daniel Shechtman observed that the diffraction pattern of an aluminum-palladium-manganese alloy had a fivefold symmetry. Daniel Shechtman could not quite believe it.





The X-ray diffraction picture of an aluminium-palladium-manganese quasicrystal.
The $2\pi/5$ symmetry of this picture contradicts the existing laws of crystallography.

X-ray crystallography

- X-ray crystallography was discovered in 1912 by Von Laue and gives a direct access to the geometry of a molecule when this molecule can crystallize.
- X-ray crystallography is a method of determining the arrangement of atoms within a crystal, in which a beam of X-rays strikes a crystal and causes the beam of light to spread into many specific directions. From the angles and intensities of these diffracted beams, a crystallographer can produce a three-dimensional picture of the density of electrons within the crystal. From this electron density, the mean positions of the atoms in the crystal can be determined, as well as their chemical bonds, their disorder and various other information.
- The X-ray crystallograph pattern of DNA was obtained by Rosalind Franklin in 1952. It is know as the B-form. Both James Watson and Francis Crick were struck by the simplicity and symmetry of this pattern. That is the way the double helix and the genetic code were discovered. Franklin died at age 37 from ovarian cancer.

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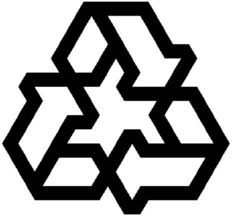
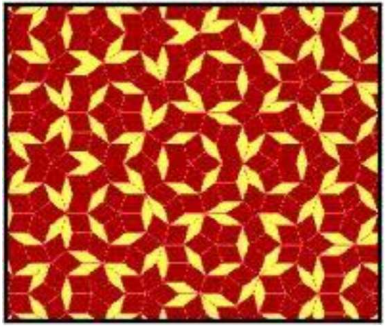
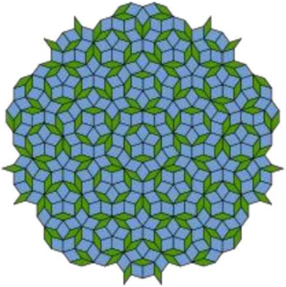
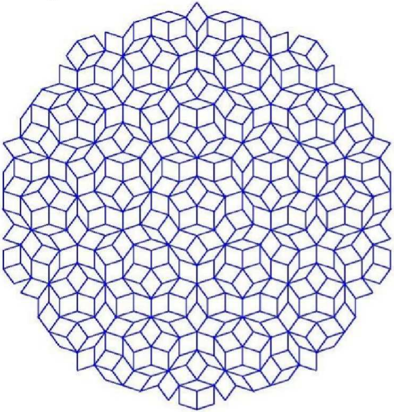


FIGURE: This picture has a $2\pi/3$ rotational symmetry.

A quasicrystal is a paving with a $2\pi/5$ symmetry.
This definition will be questioned.







The set of vertices of the rhombic Penrose tiling is a model set (N.G. de Bruijn).
Penrose, Roger (1974), Role of aesthetics in pure and applied research

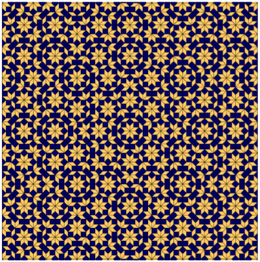


FIGURE: The set of vertices of the Ammann-Beenker tiling is a model set.

Robert Ammann (1946-1994) was an amateur mathematician who made several significant and groundbreaking contributions to the theory of quasicrystals and aperiodic tilings.

Ammann attended Brandeis University, but generally did not go to classes, and left after three years. He worked as a programmer for Honeywell. After ten years, his position was eliminated as part of a routine cutback, and Ammann ended up working as a mail sorter for a post office.

In 1975, Ammann read an announcement by Martin Gardner of new work by Roger Penrose. Penrose had discovered two simple sets of aperiodic tiles, each consisting of just two quadrilaterals. Since Penrose was taking out a patent, he wasn't ready to publish them, and Gardner's description was rather vague. Ammann wrote a letter to Gardner, describing his own work, which duplicated one of Penrose's sets, plus a foursome of "golden rhombohedra" that formed aperiodic tilings in space.

More letters followed, and Ammann became a correspondent with many of the professional researchers.

- He discovered several new aperiodic tilings, each among the simplest known examples of aperiodic sets of tiles. He also showed how to generate tilings using lines in the plane as guides for lines marked on the tiles, now called “Ammann bars".
- The discovery of quasicrystals in 1982 changed the status of aperiodic tilings and Ammann’s work from mere recreational mathematics to respectable academic research. After more than ten years of coaxing, he agreed to meet various professionals in person, and eventually even went to two conferences and delivered a lecture at each.
- Afterwards, Ammann dropped out of sight, and died of a heart attack a few years later. News of his death did not reach the research community for a few more years.

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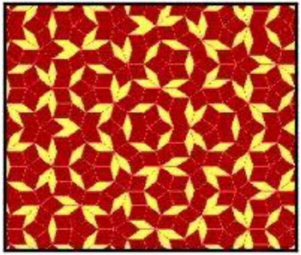
FIGURE: The physicist Peter Lu and his cousin.

The physicist Peter Lu from Harvard visited in 2007 a madrassa in Boukhara, Ouzbekistan. He is pictured here with his cousin. He realized that the tiling of the wall was indeed **a Penrose tiling with a** $2\pi/5$ **rotational symmetry**. This madrassa was constructed in the 15th century. Islamic artists designed these beautiful tilings six hundred years before western mathematicians, physicists and chemists discovered quasicrystals.

FIGURE: Boukhara (Ouzbekistan) and Darb-i Imam Ispahan (Iran).



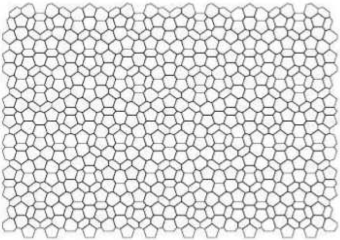


FIGURE: Paving the AlexanderPlatz in Berlin efficiently with Quasi-Periodic Tiling.

Ulrich Kortenkamp

Departament of Computer Science University of Education Schwäbische Gmünd Oberbettringer Strasse 200 73525 Schwäbisch Gmünd

Four options :

- *Tilings with a* $2\pi/5$ *symmetry* [Roger Penrose (1974)]
- *Sets of points generalizing lattices* [Y.M. (1970)]
- *"Model sets" (sets of points constructed by the "cut and project" scheme)* [Y.M. (1970)]
- *Certain Aluminium-Manganese alloys with icosahedral symmetry* [J. W. Cahn, D. Gratias et al. Phys. Rev. Lett. 53, 1951-1953 (1984)].

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A floor covered with square tiles or a floor covered with hexagonal tiles are periodic tilings. Here are a few examples of periodic tilings.

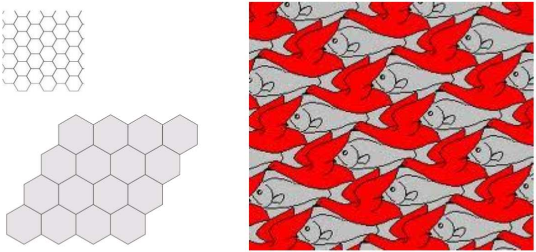

FIGURE: Paving with Quasi-Periodic Tiling.

FIGURE: One of the pavings decorating Alhambra ; Alhambra tilings revisited by M.C. Escher

Periodic tilings are repetitive.

- In repetitive music repetition is not a mere repetition of identical elements, but a repetition in another guise.
- An example is given by Ravel's Bolero. *Traditional malouf music was sometimes criticized as being* **repetitive**.
- The subtle changes which occur in malouf were not perceived by these critics. *Traditional malouf opens the gate to quasicrystals.*
- Quasicrystals are repetitive but not periodic. Repetitive tilings will be defined later on.

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The pinwheel tiling

Paving the plane with a finite number of tiles in a non periodic way is a problem which already fascinated Johannes Kepler (1571-1630). Quasicrystals are repetitive tilings. The beautiful tiling which is pictured here is *the pinwheel tiling*. It has been designed by John Conway and Charles Radin (1994).
The pinwheel tiling is NOT repetitive. The definition of a repetitive tiling will be given below.

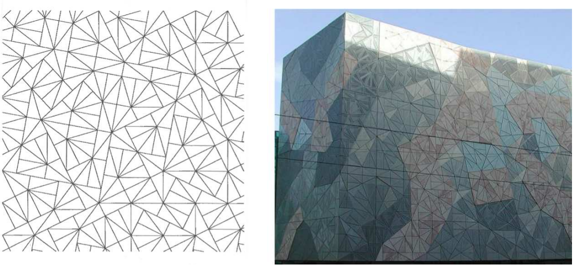


FIGURE:

Federation Square, a building complex in Melbourne, Australia features the pinwheel tiling. In the project, the tiling pattern is used to create the structural sub-framing for the façades, allowing for the façades to be fabricated off-site, in a factory and later erected to form the façades. The pinwheel tiling system was based on the single triangular element, composed of zinc, perforated zinc, sandstone or glass (known as a tile), which was joined to 4 other similar tiles on an aluminum frame, to form a “panel". Five panels were affixed to a galvanized steel frame, forming a “mega-panel", which were then hoisted onto support frames for the façade.

FIGURE: Federation Square, a building complex in Melbourne, Australia

The rotational (pinwheel) positioning of the tiles gives the façades a more random, uncertain compositional quality, even though the process of its construction is based on pre-fabrication and repetition.

To construct the pinwheel tiling, let us start with the right triangle T whose vertices are $(0,0), (2,0)$, and $(0,1)$:

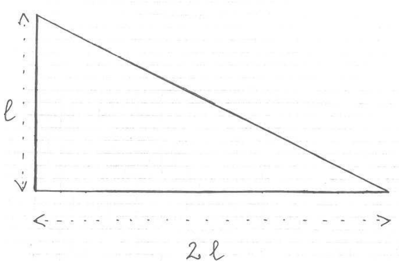
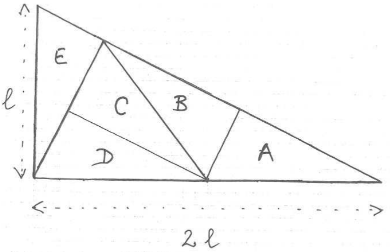


FIGURE: Divide this triangle into five isometric triangles T_1, T_2, T_3, T_4, T_5

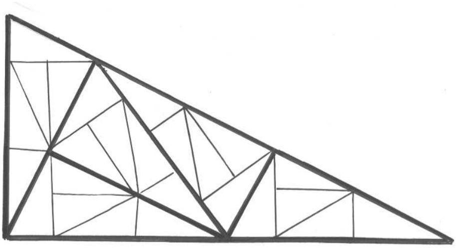
isometry :

a translation followed by a rotation or a reflection

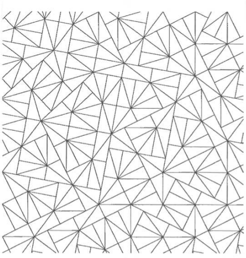
We split this right triangle T into five pieces. These pieces A, B, C, D and E are rescaled copies of T. Each piece A, B, C, D, and E is isometric to $5^{-1/2}T$. *The piece C will be named the core of T*. We iterate this decomposition on each of the pieces A, B, C, D, and E. The iterated cores T_j, $j \geq 0$, are defined by $T_0 = T$, $T_1 = C$, and T_{j+1} is the core of T_j.



We proceed and iterate this decomposition N times. Then T is decomposed into 5^N right triangles which are isometric copies of $5^{-N/2}T$. The paving of T by these 5^N right triangles is denoted by $\mathcal{P}_N$. The correponding sequence of cores T_N converge to the point (1/2, 1/2). Let us denote by S the similitude of center $(1/2, 1/2)$, of ratio $\sqrt{5}$ and of angle $\mathrm{argtan}(1/2)$. Using complex numbers this can be written $S(z) = (2+i)z - i$. We have $S(T_{N+1}) = T_N$ and $S^N(T_N) = T$.



We zoom in. We dilate the picture using the iterated similitudes S^N. The dilation factor equals $5^{N/2}$. We then have $S^N(T_N) = T$. This construction yields the increasing sequence $S^N(\mathcal{P}_N)$ of embedded pavings which converges to the pinwheel tiling $\mathcal{P}$ of the plane.

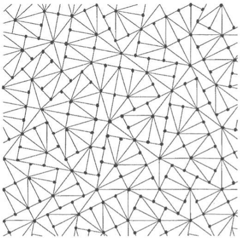


This beautiful construction was achieved by John Conway and Charles Radin.

The set Λ of all vertices in $\mathcal{P}$ will play a key role in our discussion. A vertex of the paving is simply a vertex of one of the tiles. We have

$$(2+i)\Lambda \subset \Lambda + i$$

and this self similarity preludes the properties of quasicrystals.



However Λ is not a quasicrystal. Charles Radin proved that the set of orientations of the triangles of $\mathcal{P}$ is infinite. This implies that Λ is not finitely generated.

The Delone (or Delaunay) triangulation maps a Delone set of points into a paving of the plane consisting of triangular tiles.

FIGURE: *Boris Nikolaïevitch Delaunay (Delone)*, St.Petersburg 1890- Moscow 1980.

Definition 1

A collection of points $\Lambda \subset \mathbb{R}^2$ is a Delone set if there exist two radii $R_2 > R_1 > 0$ such that

(a) every disc with radius R_1, whatever be its location, cannot contain more than one point in Λ

(b) every disc with radius R_2, whatever be its location, shall contain at least one point in Λ.

Equivalent formulation of (a) : there exists a number $\beta > 0$ such that $\forall \lambda \in \Lambda$, $\forall \lambda' \in \Lambda$,

$$\lambda \neq \lambda' \Rightarrow |\lambda' - \lambda| \geq \beta > 0. \tag{1}$$

Equivalent formulation of (b) :

$$\sup_{x \in \mathbb{R}^2} \operatorname{distance}(x, \Lambda) = \gamma < \infty \tag{2}$$

If the Delaunay triangulation is applied to a Delone set Λ, it yields a paving of the plane consisting of triangles and the set of vertices of those triangles is the Delone set Λ.
The definition of the Delaunay triangulation is explained by the following picture.

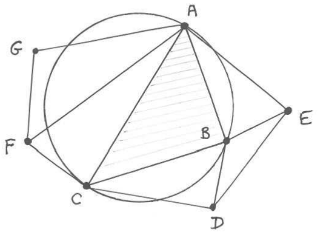


FIGURE: Delaunay triangulation.

One is given a set of points $A, B, C, D, \ldots$ The Delaunay triangulation has the property that the circumscribed circle of each triangle does not contain in its interior any other point

Local patches are efficient tools for investigating a Delone set Λ. Here is the definition of local patches. We fix a large number R and consider the moving window $Q(x,R) = [x-R, x+R] \times [x-R, x+R]$.

Definition 2

The *local patches* of the Delone set Λ are defined as

$$Q(x,R) \cap \Lambda,\ x \in \Lambda \tag{3}$$

The *centered local patches* of the Delone set Λ are defined as

$$Q(x,R) \cap \Lambda - x,\ x \in \Lambda \tag{4}$$

Local features in Λ are detected by moving x all over the set Λ and comparing the corresponding *centered local patches*.
The following pictures illustrate this definition

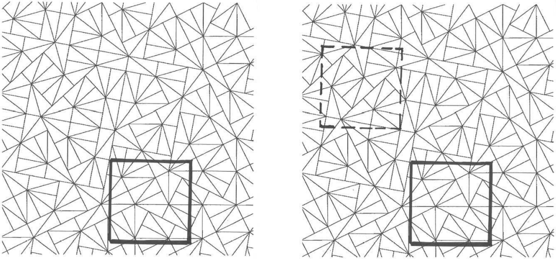


FIGURE: Local Patches.

A Delone set Λ of the plane is repetitive if every local configuration repeats itself infinitely many times in Λ. *Nietzsche's eternal recurrence.*

Definition 3

A Delone set Λ of the plane is repetitive if for every $R > 0$ there exists a $T > 0$ such that, for every $x \in \Lambda$, every disc of radius T contains a $y \in \Lambda$ such that

$$Q(y, R) \cap \Lambda - y = Q(x, R) \cap \Lambda - x \tag{5}$$

- The pinwheel tiling is NOT repetitive. Quasicrystals are repetitive.
- But a repetitive Delone set is not always a quasicrystal.
- A one dimensional example is given by $\Lambda = \cup_1^\infty \Lambda_j$ where $\Lambda_j = 2^j + r_j + 4^j\mathbb{Z}$. The sequence $r_j,\ j \geq 1$, is assumed to be dense in $[1/3, 2/3]$.
- Then Λ is a Delone set ; it is a repetitive set. But Λ is NOT an almost lattice.
  Indeed $\Lambda - \Lambda$ is not a Delone set.
- Moreover Λ is NOT finitely generated. Finally Λ is not a model set in the sense given in Theorem 2. A two dimensional example is easily constructed with the same recipe.

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- A one dimensional example is given by $\Lambda = \cup_1^\infty \Lambda_j$ where $\Lambda_j = 2^j + r_j + 4^j\mathbb{Z}$. The sequence $r_j,\ j \geq 1$, is assumed to be dense in $[1/3, 2/3]$.
- Then Λ is a Delone set ; it is a repetitive set. But Λ is NOT an almost lattice.
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Paving the plane with tiles which are isometric copies of finitely many items leads to the following definition :

Definition 4

A finitely generated paving (or tiling) $\mathcal{P}$ of the plane is a collection of subsets $T_j, j \in J$ (named tiles) which have the following properties :

(1) The plane is the union of the tiles $T_j, j \in J$.

(2) The tiles T_j are polygons and are either pairwise disjoint or only intersect at their boundaries.

(3) The tiles T_j are generated from a finite number of prototiles $P_m, 1 \leq m \leq M$ by translations and rotations.

- If only translations are used to generate the tiles form the prototyles, then $\mathcal{P}$ is a *special finitely generated paving* of the plane.

Definition 5

A *special finitely generated paving* of the plane is a Penrose paving iff it has a $2\pi/5$ rotational symmetry.

- The Conway-Radin paving is a finitely generated paving but is not a special finitely generated paving.
- The set of vertices of *special finitely generated pavings* are characterized as follows (J. Lagarias) :

Lemma 1

The following three properties of a Delone set Λ are equivalent ones

(a) For every finite set F, $\Lambda + F$ is a Delone set.

(b) For each $R > 0$, there are only finitely many R−patches $\Lambda_{y,R}$, $y \in \Lambda$.

(c) There are only finitely many R−patches $\Lambda_{y,R}$ when $y \in \Lambda$ and $R = 2R_2$.

- The constant R_2 in (c) is defined by property (b) of Definition 1.
- There are finitely many local configurations in Λ only.
- Then the Delone triangulation applied to Λ yields the familiar picture of a Penrose paving. The tiles which are used in the paving are translated of a finite set of tiles.

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- Almost lattices are sets $\Lambda \subset \mathbb{R}^n$ of points which generalize lattices.

Definition 6

An almost lattice Λ is a Delone set such that $\Lambda - \Lambda \subset \Lambda + F$ where F is a finite set.

- We begin with a counter example. The set Λ of vertices of the pinwheel tiling is not an almost lattice. Indeed Λ does not even satisfy the equivalent properties of Lemma 1.
- Let us now give an example. Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Let $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$ be its conjugate. Consider the number field $\mathcal{K} = \mathbb{Q}(\sqrt{5})$. Then the algebraic integers in $\mathcal{K}$ are $x = m + n\varphi, \ m, n \in \mathbb{Z}$. The set of these algebraic integers is a ring denoted by $\mathcal{O}$. The conjugate $\overline{x}$ of $x \in \mathcal{O}$ is $\overline{x} = m + n\overline{\varphi}$. We now consider the set $S \subset \mathbb{R}$ of all $x \in \mathcal{O}$ such that $|\overline{x}| \leq 1$. Then S is an almost lattice.

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- F. Lagarias proved the following theorem

Theorem 1

A Delone set Λ is an almost lattice if and only if $\Lambda - \Lambda$ is a Delone set.

- If $F = \{0\}$, Λ is a lattice. Almost lattices are named “Meyer sets" in the literature. If $M \subset \mathbb{Z}$ is a set of integers, then M is an almost lattice if and only if the distance between two consecutive integers in M is bounded.
- For example let us define M as the collection of $7k + r_k,\ k \in \mathbb{Z}$, where the only condition which is imposed on r_k is $r_k \in \{0, 1, \dots, 6\}$. Then M is always an almost lattice.

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An almost lattice can be irregular. An almost lattice is not almost periodic in any sense. Model sets which are defined now are more regular point sets.

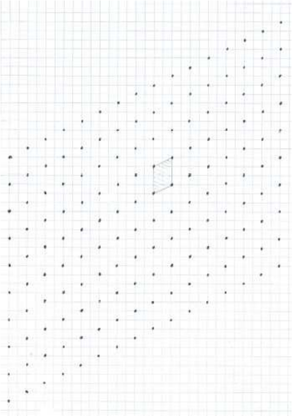

FIGURE: Lattice .

- *Model sets were first studied systematically in 1972 by Yves Meyer, who considered them in the context of Diophantine problems in harmonic analysis. More recently, model sets have played a prominent role in the theory of quasicrystals, beginning with N. G. de Bruijn's 1981 discovery that the vertices of a Penrose tiling are a model set.*
- *Much of the interest in model sets is due to the fact that although they are aperiodic, model sets have enough "almost periodicity" to give them a discrete Fourier transform. This corresponds to spots, or Bragg peaks, in the X-ray diffraction pattern of a quasicrystal. (E. Arthur Robinson Jr.)*
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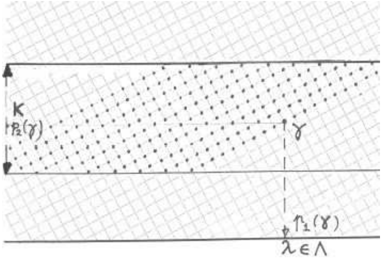
Let, as above, $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio.

We consider the lattice Γ of the plane consisting of all $x = (x_1, x_2)$ where

(6) $$x_1 = m + n\varphi,\ x_2 = m + n\overline{\varphi},\quad m, n \in \mathbb{Z}$$

Then the set S consisting of all such x_1 such that $|x_2| \leq 1$ is a model set.



Cut and Projection.

Γ is a lattice, $\gamma \in \Gamma$, $\gamma = (p_1(\gamma), p_2(\gamma))$.
$p_1 : \Gamma \to p_1(\Gamma)$ is a one-to-one mapping and $p_2(\Gamma)$ is a dense subgroup of $\mathbb{R}$.
$\Lambda = \{\lambda = p_1(\gamma);\ \gamma \in \Gamma,\ p_2(\gamma) \in K\}$

- We have

Theorem 2

A model set is an almost lattice. Conversely if Λ is an almost lattice there exists a model set M and a finite set F such that $\Lambda \subset M + F$.

- A lattice $\Gamma \subset \mathbb{R}^N$ is a discrete subgroup with compact quotient. In other words $\Gamma = A(\mathbb{Z}^N)$ where A is an $N \times N$ invertible matrix.
- One starts with an integer $m \geq 1$, we set $N = n + m$, $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and consider a lattice $\Gamma \subset \mathbb{R}^N$. If $(x, y) = X \in \mathbb{R}^n \times \mathbb{R}^m$, we write $x = p_1(X)$ and $y = p_2(X)$.
- Let us assume that $p_1 : \Gamma \to p_1(\Gamma)$ is a one-to-one mapping and that $p_2(\Gamma)$ is a dense subgroup of $\mathbb{R}^m$.

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- A set $K \subset \mathbb{R}^m$ is Riemann integrable if its boundary has a zero Lebesgue measure. The boundary of K is $\overline{K} \setminus L$ where $\overline{K}$ is the closure of K and L is the interior of K. The interior of K is the largest open set contained in K. If K is Riemann integrable, then K has a positive measure if and only if K has a non-empty interior.

Definition 7

Let K be a Riemann integrable compact subset of $\mathbb{R}^m$ with a positive measure. Then the model set Λ defined by Γ and K is

$$\Lambda = \{\lambda = p_1(\gamma);\, \gamma \in \Gamma,\, p_2(\gamma) \in K\} \tag{7}$$

A subset Λ of $\mathbb{R}^n$ is a model set if either Λ is a lattice or if one can find m, Γ, and K such that Λ is the model set defined by (7).

- The compact set K is named the *window* of the model set Λ.

Lemma

Let ∂K be the frontier of K. Then any model set Λ for which $\partial K \cap p_2(\Gamma) = \emptyset$ is repetitive.

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A Pisot-Vijayaraghavan number is a real number $\theta > 1$ with the following two properties :

(a) θ is an algebraic integer of degree $n \geq 1$

(b) the $n-1$ conjugates $\theta_2, \dots, \theta_n$ of θ satisfy

$$|\theta_2| < 1, \dots, |\theta_n| < 1. \tag{8}$$

For example, the natural integers $2, 3, \dots$ are Pisot-Vijayaraghavan numbers and condition (b) is vacuous in that case. When the degree n of a Pisot number θ exceeds 1, the minimal polynomial of θ is $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ where $a_1 \in \mathbb{Z}, \dots, a_n \in \mathbb{Z}$. Then the conjugates $\theta_2, \dots, \theta_n$ of θ are the other solutions to $P(z) = 0$ and can be either real or complex numbers.

- The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is a Pisot number. The minimal polynomial of φ is $x^2 - x - 1$ and the conjugate of φ is $\frac{1-\sqrt{5}}{2}$. The golden ratio φ is not the smallest Pisot number. The smallest Pisot number $\rho = 1.324717\ldots$ is named the plastic number and is the real solution to the equation $x^3 - x - 1 = 0$. The two other solutions z_1 and z_2 to this equation are complex numbers. We have $z_1 = \overline{z}_2$ and $z_1 z_2 = |z_1|^2 = |z_2|^2 = 1/\rho$ which is fully consistent with the fact that ρ is a Pisot number. Raphaël Salem proved that the set S of all Pisot numbers is closed.
- Salem numbers are defined the same way. On keeps condition (a) but replaces (b) by $|\theta_2| \leq 1, \ldots, |\theta_n| \leq 1$ with, at least, equality somewhere. Then the degree n of θ is even. Up to some permutation between the conjugates we always have $\theta_2 = \frac{1}{\theta}$ and $|\theta_3| = \cdots = |\theta_n| = 1$.

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Before stating our main result, let us provide the reader with an example. Let φ be the golden ratio. We define $S \subset \mathcal{O}$ as above. Then we have

$$\varphi S \subset S.$$

The proof is trivial. If $x \in S$, then $x \in \mathcal{O}$ and $|\overline{x}| \leq 1$. The conjugate of the product φx is the product of the conjugates which concludes the proof.

Theorem 3

Let $\Lambda \subset \mathbb{R}^n$ be a model set. If $\theta > 1$ and $\theta\Lambda \subset \Lambda$, then θ is either a Pisot number or a Salem number.

Conversely for each dimension n and each Pisot or Salem number θ, there exists a model set $\Lambda \subset \mathbb{R}^n$ such that $\theta\Lambda \subset \Lambda$

- The following theorem explains Daniel Shechtman's fundamental discovery.
- Let Λ be a model set defined as above by a lattice $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ and a compact set $K \subset \mathbb{R}^m$. We let H denote the group $p_1(\Gamma^*)$ where Γ^* is the dual lattice of Γ. Let us assume K to be Riemann-integrable with a positive measure and let φ denote any $C_0^\infty(\mathbb{R}^m)$ function vanishing outside K.
- The corresponding weight factors $w(\lambda)$, $\lambda \in \Lambda$, are defined on the model set Λ by

$$w(p_1(\gamma)) = \varphi(p_2(\gamma)), \ \gamma \in \Gamma$$

If φ was the indicator function of K (this indicator function is not smooth), we would have $w(\lambda) = 1$ on Λ.

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With these notations, one obtains

Theorem 4

Let μ be the sum

$$\sum_{\lambda \in \Lambda} w(\lambda)\delta_\lambda$$

of Dirac masses over Λ where the weight factors $w(\lambda)$ are defined as above. Then the distributional Fourier transform of μ is the atomic measure ν defined by

$$\nu = \frac{(2\pi)^n}{\mathrm{vol}\,\Gamma} \sum_{\gamma^* \in \Gamma^*} \hat{\varphi}(-(p_2(\gamma^*))\, \delta_{p_1(\gamma^*)}$$

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