Translation surfaces and their geodesics (II)

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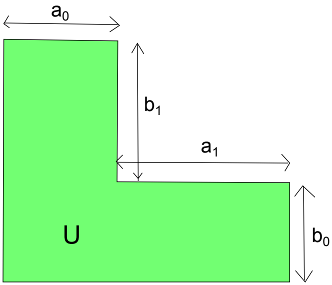
The setting

We will consider the billiards dynamics in a *L*-shaped table *U* depending on parameters $a_0, a_1, b_0, b_1 > 0$.



Constructing a translation surface from the billiards table

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We set $s_O(\theta) := \pi + \theta$. We have $s_O = s_h \circ s_v = s_v \circ s_h$. If a trajectory starts at time 0 in the direction $\theta(0) = \theta_0$, its direction $\theta(t)$ at any time t can only take one of the values $\pm\theta_0$, $\pi \pm \theta_0$.

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We define the linear symmetries S_h, S_v of $\mathbb{R}^2$ associated to s_h, s_v , and their composition $S_O = S_h \circ S_v = S_v \circ S_h$. The linear maps $\mathrm{id}, S_h, S_v, S_O$ form a group G (the Klein group).

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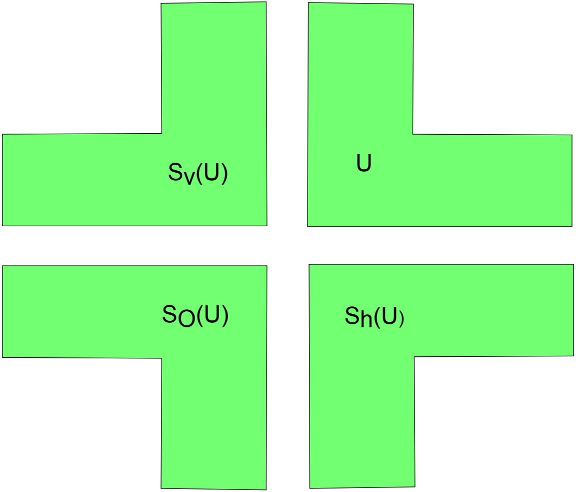
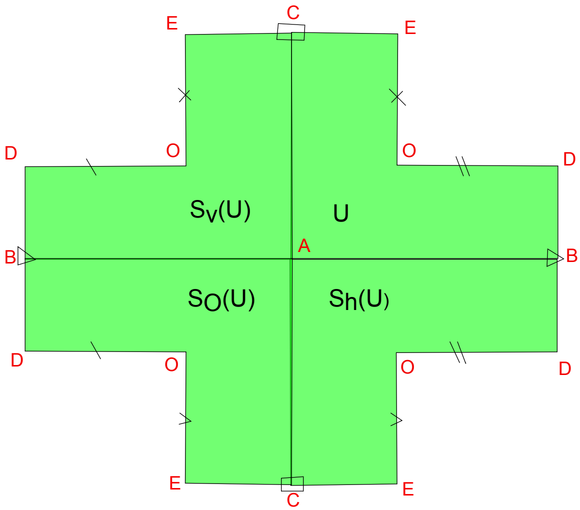
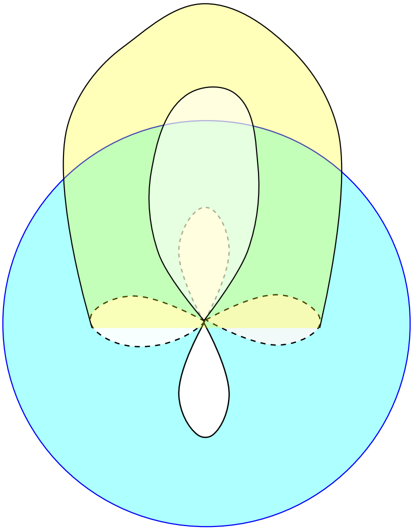
For any $g \in G$, any horizontal side C of $g(U)$ is glued through S_h to the side $S_h(C)$ of $S_h \circ g(U)$, and any vertical side C of $g(U)$ is glued through S_v to the side $S_v(C)$ of $S_v \circ g(U)$.



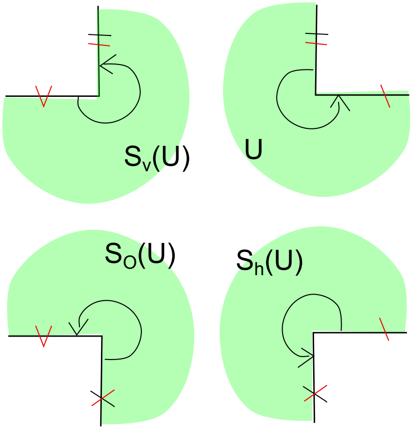
The four copies before glueing



Parallel sides with the same label must still be identified. Vertices with the same name correspond to the same point on M.



Attaching a handle to a sphere



The local picture at the special point O

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- From the topological point of view, M is a sphere with two handles attached . One says that M is a *surface of genus 2*. The torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ is a surface of genus 1.
- The total angle around A, B, C, D, E is 2π, but the total angle around O is 6π. Any point of M except O has a natural local system of coordinates , well-defined up to translation.

Linear flows on M

Let u, v be real parameters. The differential equation

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v,$$

(where (x, y) is any system of natural local coordinates on $M - \{O\}$), defines a flow $\Phi^t_{u,v}$ on M.

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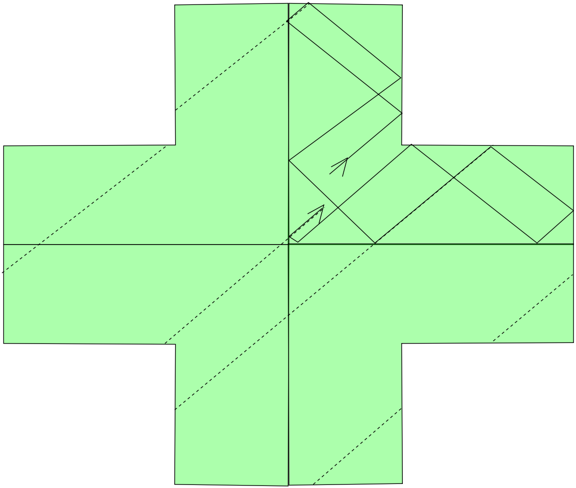
As in the rectangular case, the direction $\theta(t)$ at time t and the position $(x(t), y(t))$ are determined by the position $\Phi^t_{u,v}(x_0, y_0)$ in M.



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Exercise: Assume that the parameters a_0, a_1, b_0, b_1 of the table U are rational. Then a direction θ is M-rational iff $\tan\theta \in \mathbb{Q} \cap \infty$.

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Theorem: (Keane)
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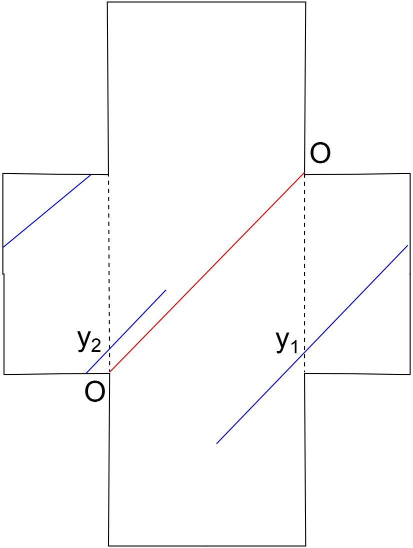
O
y2
y1
O

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$$N(p, \theta, T) := (h_0(p, \theta, T), h_1(p, \theta, T), v_0(p, \theta, T), v_1(p, \theta, T)) \in \mathbb{Z}^4.$$

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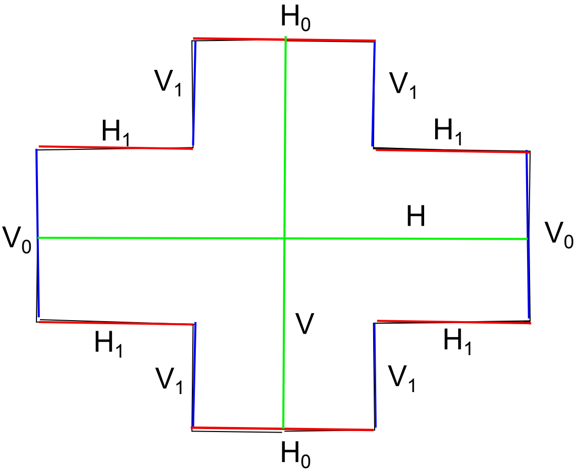
A similar inequality holds for the number of hits on the large vertical side.



The closed geodesic loops on M associated to the sides of the table U

Heuristics on expected hitting statistics

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- One "expects" (uniform distribution of intersections with H) that the probability of hitting H_0 or H_1 is proportional to the length, respectively $\frac{a_0}{a_0+a_1}$ and $\frac{a_1}{a_0+a_1}$.

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- The expected sizes of $h_0(p,\theta,T)$ and $h_1(p,\theta,T)$ are thus $|\sin\theta|\frac{a_0}{2S}T$ and $|\sin\theta|\frac{a_1}{2S}T$ respectively.

Uniform distribution property

According to the previous heuristics, one introduces the following

Definition: A M-irrational direction θ has the *uniform distribution property* if any billiards trajectory with initial direction θ not running into the vertex O satisfies the expected statistics

$$\lim_{T\to\infty} \frac{1}{T} N(p,\theta,T) = \frac{1}{2S}(|\sin\theta|a_0, |\sin\theta|a_1, |\cos\theta|b_0, |\cos\theta|b_1).$$

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Theorem: (Masur, Veech) For any parameters a_0, a_1, b_0, b_1, almost all directions have the uniform distribution property.

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- when the irrational direction is very well approximated by rational directions (the Liouville case), one cannot improve significantly on $o(T)$;
- on the other hand, for almost all directions (the diophantine case), one can obtain the much better estimate $o(T^{\epsilon})$, for any $\epsilon > 0$.

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- For almost all parameters a_0, a_1, b_0, b_1 and almost all directions θ, one has, for any $p \in U$

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- For any rational parameters a_0, a_1, b_0, b_1 and almost all directions θ, one has, for any $p \in U$

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- For $a_0, a_1, b_0, b_1, \theta$ as above, there exists a 2-dimensional plane $P := P(a_0, a_1, b_0, b_1, \theta)$ in $\mathbb{R}^4$ containing the line $\mathbb{R}\ell$ (ℓ being the limit of $\frac{1}{T}N(p, \theta, T)$ given above) such that the distance of $N(p, \theta, T)$ to P stays $o(T^\epsilon)$, for any $\epsilon > 0$.

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Unfortunately, there is no "elementary" proof of these results at this moment.

Thanks for your attention