A 125th birthday party...

Ken Ono
Emory University

Travel back in time...

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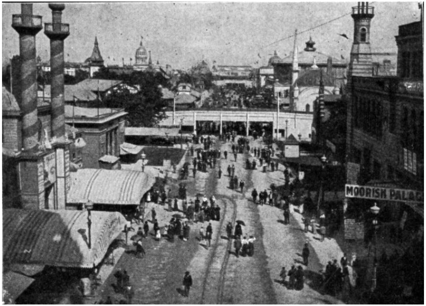
1893: Chicago hosts the **World's Fair!**



Celebrating the **state of the art in science and technology.**

Visitors enjoyed

Visitors enjoyed

- The first Ferris wheel. 

Visitors enjoyed

- The first Ferris wheel.

- Moving pictures.

Visitors enjoyed

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- Hershey's chocolate.

Visitors enjoyed

- The first Ferris wheel.

- Moving pictures.

- Hershey's chocolate.

- The first *Congress of Mathematicians*.

India and the 1893 World's Fair

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- There were no exhibits from India.

India and the 1893 World's Fair

- There were no exhibits from India.

- There were no talks by Indian mathematicians.

However, in South India. . .

. . . the incredible story of Srinivasa Ramanujan was beginning...

The legend. . .

The legend. . .

- Ramanujan was born in 1887.

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- He was a Brahmin, a member of India's priestly caste.

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- He was the son of a cloth merchant.

The legend. . .

- Ramanujan was born in 1887.
- He was a Brahmin, a member of India's priestly caste.
- He was the son of a cloth merchant.
- He was an excellent student, earning a scholarship to college.

A turning point

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- In college a friend introduced him to G. S. Carr's **Synopsis of elementary results in pure mathematics.**

A turning point

- In college a friend introduced him to G. S. Carr's

  Synopsis of elementary results in pure mathematics.

  "the 'synopsis' it professes to be. It contains enunciations of 6165 theorems, systematically and quite scientifically arranged, with proofs which are often little more than cross-references..."

Ramanujan's new found infatuation.

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- Imitating Carr, he recorded his findings in notebooks...

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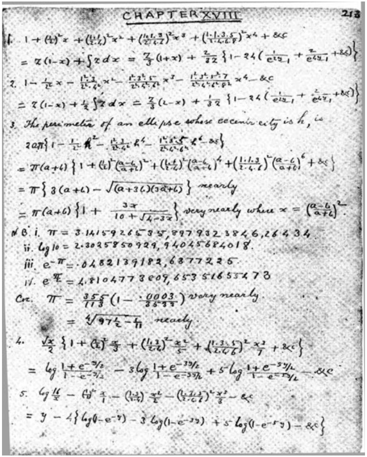

CHAPTER XVIII 213

1. $1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.1}{2.4}\right)^2 x^2+\left(\frac{1.1.3}{2.4.6}\right)^2 x^3+\left(\frac{1.1.3.5}{2.4.6.8}\right)^2 x^4+\&c$

$= z(1-x)+\int z\,dx = \frac{z}{3}(1+x)+\frac{2}{3z}\left\{1-24\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)\right\}$

2. $1-\frac{1}{2^2}x-\frac{1^2.3}{2^2.4^2}x^2-\frac{1^2.3^2.5}{2^2.4^2.6^2}x^3-\frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8^2}x^4-\&c$

$= z(1-x)+\frac{1}{2}\int z\,dx = \frac{z}{3}(2-x)+\frac{1}{3z}\left\{1-24\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)\right\}$

3. The perimeter of an ellipse whose eccentricity is k, is

$2a\pi\left\{1-\frac{1}{2^2}k^2-\frac{1^2.3}{2^2.4^2}k^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}k^6-\&c\right\}$

$= \pi(a+b)\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{a-b}{a+b}\right)^2+\left(\frac{1.1}{2.4}\right)^2\left(\frac{a-b}{a+b}\right)^4+\left(\frac{1.1.3}{2.4.6}\right)^2\left(\frac{a-b}{a+b}\right)^6+\&c\right\}$

$= \pi\left\{3(a+b)-\sqrt{(a+3b)(3a+b)}\right\}$ nearly

$= \pi(a+b)\left\{1+\frac{3x}{10+\sqrt{4-3x}}\right\}$ very nearly where $x=\left(\frac{a-b}{a+b}\right)^2$.

N.B. i. $\pi = 3.14159265358,97932384 6,26434$

ii. $\log 10 = 2.302585092 9,9404568401 8$.

iii. $e^{-\pi} = .0432139182,6377225$

iv. $e^{\frac{\pi}{2}} = 4.8104773809,6535165547 3$

Cor. $\pi = \frac{355}{113}\left(1-\frac{.0003}{3533}\right)$ very nearly.

$= \sqrt[4]{97\frac{1}{2}-\frac{1}{11}}$ nearly.

4. $\frac{\sqrt{x}}{2}\left\{1+\left(\frac{1}{2}\right)^2\frac{x}{3}+\left(\frac{1.3}{2.4}\right)^2\frac{x^2}{5}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{x^3}{7}+\&c\right\}$

$= \log\frac{1+e^{-y/2}}{1-e^{-y/2}}-3\log\frac{1+e^{-3y/2}}{1-e^{-3y/2}}+5\log\frac{1+e^{-5y/2}}{1-e^{-5y/2}}-\&c$

5. $\log\frac{16}{x}-\left(\frac{1}{2}\right)^2\frac{x}{1}-\left(\frac{1.3}{2.4}\right)^2\frac{x^2}{2}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{x^3}{3}-\&c$

$= y-4\left\{\log(1-e^{-y})-3\log(1-e^{-3y})+5\log(1-e^{-5y})-\&c\right\}$

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- He gave no proofs of any kind.
- Ramanujan lost interest in everything but math.

  And he flunked out of college... **Twice!**

Mathematical Purgatory

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- He found work as a clerk at the Madras Port Trust.

Mathematical Purgatory

- To their credit, his parents continued to support him.
- He found work as a clerk at the Madras Port Trust.
- He continued to work at his math, scribbling madly on a heavy slate and in his prized notebooks.

Letter to Hardy

- After years in isolation and seeking recognition,

Letter to Hardy

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G. H. Hardy, Sadlierian Professor of Mathematics
Cambridge University

Hardy invited Ramanujan to Cambridge.

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Hardy invited Ramanujan to Cambridge.

- At first Ramanujan declined for religious reasons.
- Visions from Goddess Namagiri granted him permission.

Ramanujan in England

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- He spent the next 5 years in England.

Ramanujan in England

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Published over 30 papers:

- Prime numbers.
- Hypergeometric series.
- Elliptic functions.
- Partitions.
- Probabilistic Number Theory

Glory and Tragedy

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- Elected Fellow of the Royal Society.

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- Ramanujan achieved this despite the hardships of WWI.
- Ramanujan grew ill in 1919, and returned to India.
- Ramanujan died in Madras on April 26, 1920.

Ramanujan's Legacy

Fields Medals have been awarded for solving his problems.

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- Ramanujan graphs
- "Circle method" in Analytic Number Theory.
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- Probabilistic Number Theory
- and the list goes on and on...

Adding and Counting

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Question. In how many ways can 4 be written as sum?

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$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1,$$

Adding and Counting

Question. In how many ways can 4 be written as sum?

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1,$$

We say that $p(4) = 5$.

The Partition function $p(n)$

Definition

A **partition** of an integer n is any nonincreasing sequence of positive integers which sum to n.

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A **partition** of an integer n is any nonincreasing sequence of positive integers which sum to n.

Notation. The partition function

$$p(n) = \text{Number of partitions of } n.$$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

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Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$
- $p(8) =$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$
- $p(8) = 22$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$
- $p(8) = 22$
- $p(16) =$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$
- $p(8) = 22$
- $p(16) = 231$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$
- $p(8) = 22$
- $p(16) = 231$
- $p(32) =$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$
- $p(8) = 22$
- $p(16) = 231$
- $p(32) = 8349$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$
- $p(8) = 22$
- $p(16) = 231$
- $p(32) = 8349$
- $p(64) =$

Is there a simple formula for $p(n)$?

Here are some values of $p(n)$:

- $p(2) = 2$
- $p(4) = 5$
- $p(8) = 22$
- $p(16) = 231$
- $p(32) = 8349$
- $p(64) = 1741630$

Hardy-Ramanujan Formula

Hardy-Ramanujan Formula

Theorem (Hardy and Ramanujan)

We have that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

The Hardy-Ramanujan Formula

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n	$p(n)$	HR Formula	$\frac{p(n)}{\text{HR Formula}}$

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n	$p(n)$	HR Formula	$\frac{p(n)}{\text{HR Formula}}$
10	42	$48.10\ldots$	$0.87\ldots$

The Hardy-Ramanujan Formula

n	$p(n)$	HR Formula	$\frac{p(n)}{\text{HR Formula}}$
10	42	$48.10\ldots$	$0.87\ldots$
20	627	$692.38\ldots$	$0.90\ldots$

The Hardy-Ramanujan Formula

n	$p(n)$	HR Formula	$\frac{p(n)}{\text{HR Formula}}$
10	42	48.10...	0.87...
20	627	692.38...	0.90...
$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	190,569,292	199,280,893.34...	0.95...

The Hardy-Ramanujan Formula

n	$p(n)$	HR Formula	$\frac{p(n)}{\text{HR Formula}}$
10	42	$48.10\ldots$	$0.87\ldots$
20	627	$692.38\ldots$	$0.90\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	190,569,292	$199{,}280{,}893.34\ldots$	$0.95\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
100,000	Large #	Large #	$0.998\ldots$

Divisibility of $p(n)$

The beginning of a pattern:

Divisibility of $p(n)$

The beginning of a pattern:

- $p(4) = 5$

Divisibility of $p(n)$

The beginning of a pattern:

- $p(4) = 5$
- $p(9) = 30$

Divisibility of $p(n)$

The beginning of a pattern:

- $p(4) = 5$
- $p(9) = 30$
- $p(14) = 135$

Divisibility of $p(n)$

The beginning of a pattern:

- $p(4) = 5$
- $p(9) = 30$
- $p(14) = 135$
- $p(19) = 490$
- $p(24) = 1575$
- $p(29) = 4565$
- $p(34) = 12310$
- $\vdots \qquad \vdots$

Does the pattern continue on and on?

Does the pattern continue on and on?

Theorem (Ramanujan)

For every n, we have

$$p(5n+4) \text{ is a multiple of } 5.$$

Ramanujan's congruences

Theorem (Ramanujan)

For every n, we have that

$$\begin{aligned} p(5n+4) &\text{ is a multiple of } 5,\\ p(7n+5) &\text{ is a multiple of } 7,\\ p(11n+6) &\text{ is a multiple of } 11. \end{aligned}$$

The function $f(n)$

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Definition

Define the "first digit" function $f(n)$ by

$$f(n) := \text{“first digit of } p(n)\text{”}$$

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For example, we have

$$p(10) = 42 \longrightarrow f(10) = 4,$$

$$p(20) = 627 \longrightarrow f(20) = 6,$$

$$p(30) = 5604 \longrightarrow f(30) = 5,$$

$$p(40) = 37338 \longrightarrow f(40) = 3.$$

A natural question.

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Question

What is the "frequency" of the possible 9 values of $f(n)$*?*

A natural question.

Question

What is the "frequency" of the possible 9 values of $f(n)$?

For example, does each possible value occur with frequency $1/9$?

A natural question.

Question

What is the "frequency" of the possible 9 values of $f(n)$?

For example, does each possible value occur with frequency 1/9?

Definition (Frequency Function)

If $b \in \{1, 2, \ldots, 9\}$, then let

$$F_b(X) := \text{ Percentage of } \{n < X \ : \ f(n) = b\}.$$

Data (Percentages)

X	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9

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X	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
10	40	20	20	0	10	0	10	0	0

Data (Percentages)

X	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
10	40	20	20	0	10	0	10	0	0
20	35	20	15	10	10	0	10	0	0

Data (Percentages)

X	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
10	40	20	20	0	10	0	10	0	0
20	35	20	15	10	10	0	10	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	33	16	14	9	7	6	7	5	3

Data (Percentages)

X	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
10	40	20	20	0	10	0	10	0	0
20	35	20	15	10	10	0	10	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	33	16	14	9	7	6	7	5	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1000	30.6	17.6	12.7	9.4	7.6	6.8	5.7	5.2	4.4

Data (Percentages)

X	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
10	40	20	20	0	10	0	10	0	0
20	35	20	15	10	10	0	10	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	33	16	14	9	7	6	7	5	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1000	30.6	17.6	12.7	9.4	7.6	6.8	5.7	5.2	4.4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2500	30.2	17.8	12.4	9.6	7.7	6.7	5.7	5.0	4.6

What is going on?

Question

Do we recognize the numbers

$$30.2,\quad 17.8,\quad 12.4,\quad 9.6,\quad 7.7,\quad 6.7,\quad 5.7,\quad 5.0,\quad 4.6?$$

The theorem

The theorem

Theorem (Anderson, Rolen, Stoehr)

If $F_b := \lim_{X \to +\infty} F_b(X)$, *then*

$$F_b = \begin{cases} 30.1\% & \text{if } b = 1, \\ 17.6\% & \text{if } b = 2, \\ 12.4\% & \text{if } b = 3, \\ 9.69\% & \text{if } b = 4, \\ 7.91\% & \text{if } b = 5, \\ 6.69\% & \text{if } b = 6, \\ 5.79\% & \text{if } b = 7, \\ 5.11\% & \text{if } b = 8, \\ 4.57\% & \text{if } b = 9. \end{cases}$$

Why is this theorem true?

$$
\begin{aligned}
\log_{10}(2) - 0 &= 0.3010\ldots \\
\log_{10}(3) - \log_{10}(2) &= 0.176\ldots \\
\log_{10}(4) - \log_{10}(3) &= 0.124\ldots \\
\log_{10}(5) - \log_{10}(4) &= 0.0969\ldots \\
\log_{10}(6) - \log_{10}(5) &= 0.0791\ldots \\
\log_{10}(7) - \log_{10}(6) &= 0.0669\ldots \\
\log_{10}(8) - \log_{10}(7) &= 0.0579\ldots \\
\log_{10}(9) - \log_{10}(8) &= 0.0511\ldots \\
\log_{10}(10) - \log_{10}(9) &= 0.0457\ldots.
\end{aligned}
$$

Why is this theorem true?

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- Consider $p(32) = 8349$.

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$$p(32) = 8.349 \times 10^3.$$

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- Therefore, we find that

$$\log_{10}(p(32)) = \log_{10}(8.349) + \log_{10}(10^3) = \log_{10}(8.349) + 3.$$

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- Consider $p(32) = 8349$.
- Writing in scientific notation we get:

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- Therefore, we find that

$$\log_{10}(p(32)) = \log_{10}(8.349) + \log_{10}(10^3) = \log_{10}(8.349) + 3.$$

- Ignore the 3, and let $p^*(32) = \log_{10}(8.349) = 0.9216\ldots$.

Why is this theorem true?

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- For every $p(n)$ we get $0 < p^*(n) < 1$.

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$$\log_{10}(2) \le p^*(n) < \log_{10}(3),$$

Why is this theorem true?

- For every $p(n)$ we get $0 < p^*(n) < 1$.
- The first digit is 1 only when $p^*(n) < \log_{10}(2) = 0.3010\ldots$.
- The first digit is 2 only when

$$\log_{10}(2) \leq p^*(n) < \log_{10}(3),$$

and so on. . .

Why is this theorem true?

Why is this theorem true?

- Notice the "uneven" plot of

$$0, \log_{10}(2), \ldots, \log_{10}(9), 1.$$

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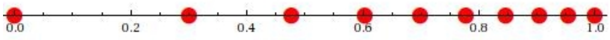

- (Benford's Law): Imagine throwing "darts".

Why is this theorem true?

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- Ramanujan's asymptotic gives precise information on $p(n)$, and consequently $p^*(n)$.

Why is this theorem true?

- Ramanujan's asymptotic gives precise information on $p(n)$, and consequently $p^*(n)$.
- Weyl gave a “randomness criterion”, which we can now verify.

Ramanujan's Legacy for Adding and Counting

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- The "size" and rapid growth of $p(n)$.

Ramanujan's Legacy for Adding and Counting

- The "size" and rapid growth of $p(n)$.
- The divisibility properties of $p(n)$.

Ramanujan's Legacy for Adding and Counting

- The "size" and rapid growth of $p(n)$.
- The divisibility properties of $p(n)$.
- The phenomenon of "first digits".

Ramanujan: The Legend

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