Translation surfaces and their geodesics (I)

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Billiards in planar domains

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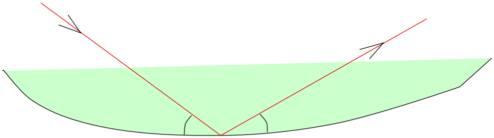
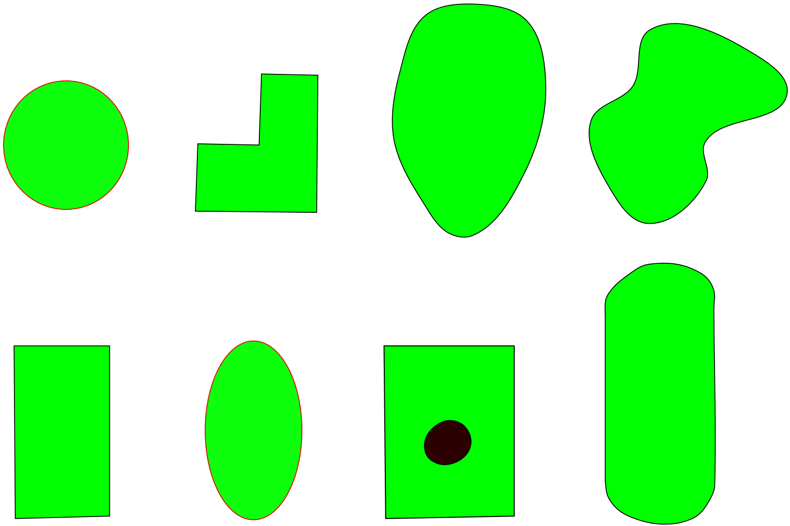
A particle runs straightforward at unit speed in U, bouncing elastically on (the smooth part of) the boundary. The motion stops if the particle hits a non regular point of the boundary.



Some interesting tables



Time averages of observables (Birkhoff averages)

Denote by $q(t) = (x(t), y(t)) \in \bar{U}$ be the position of the particle at time t, by $\theta(t) \in \mathbb{R}/2\pi\mathbb{Z}$ its direction at a non-bouncing time t (the angle being counted from the horizontal).

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Given a "nice" function $\varphi(q, \theta)$ on $\bar{U} \times \mathbb{R}/2\pi\mathbb{Z}$, we would like to understand the behaviour of the *Birkhoff averages*

$$\frac{1}{T} \int_0^T \varphi(q(t), \theta(t))\, dt$$

as T becomes large, for every initial condition $(q(0), \theta(0))$.

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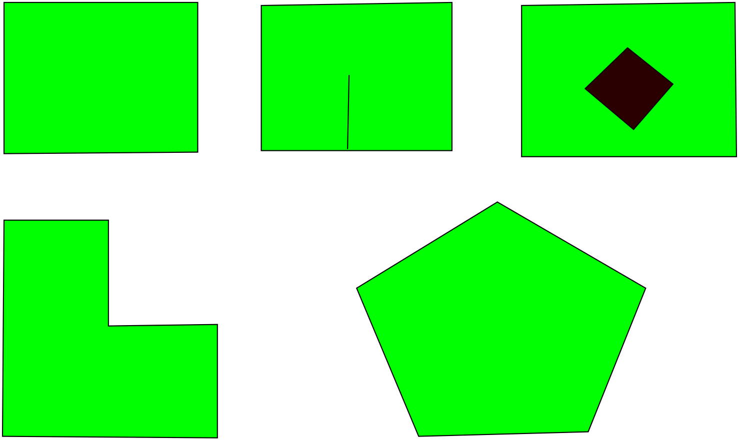
A polygonal billiards table is *rational* is any angle between the segments in the boundary is a rational multiple of 2π.

Some rational tables



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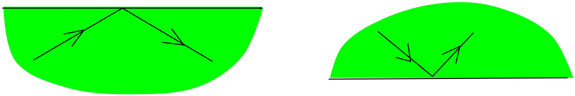

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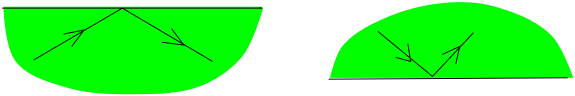
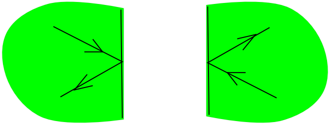


If the rebound occurs on the vertical sides of U, one has $\theta_{out} = \pi - \theta_{in} =: s_v(\theta_{in})$.



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Denote by $S_h(x, y) = (x, -y)$ and $S_v(x, y) = (-x, y)$ the linear symmetries of $\mathbb{R}^2$ associated to s_h, s_v, and by $S_O(x, y) = (-x, -y)$ the central symmetry equal to $S_h \circ S_v = S_v \circ S_h$.

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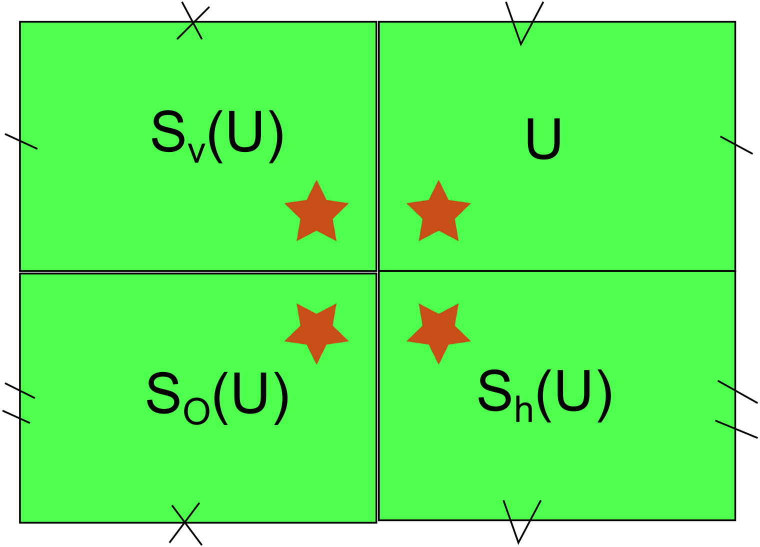
From the table U and its symmetric copies $S_h(U)$, $S_v(U)$, $S_O(U)$, we construct a closed surface in the following way.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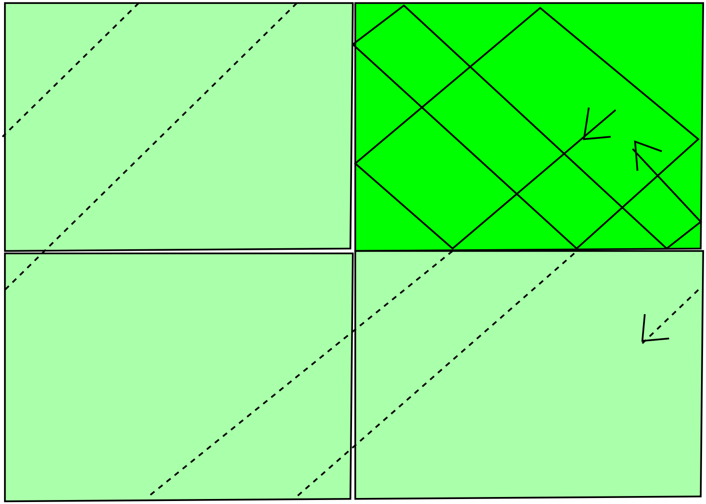
Exercise: Prove that the space obtained in this way is naturally identified with the quotient space $\mathbb{T}_{a,b} := \mathbb{R}^2/2a\mathbb{Z} \oplus 2b\mathbb{Z}$.

Such a quotient of the plane by a lattice is called a (2-dimensional) **flat torus**.

From billiards trajectories to linear flows on the torus



Linear flows on tori

Given parameters $\widetilde{u}, \widetilde{v} \in \mathbb{R}$, one defines a flow (called a **linear flow**) on $\mathbb{T}_{a,b} := \mathbb{R}^2/2a\mathbb{Z} \oplus 2b\mathbb{Z}$ by the formula

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For $\widetilde{u}, \widetilde{v} \in \mathbb{R}$, set $u := \frac{\widetilde{u}}{2a}, v := \frac{\widetilde{v}}{2b}$. The map h *conjugates* the flow $\Phi^t_{\widetilde{u},\widetilde{v}}$ on $\mathbb{T}_{a,b}$ to the flow $\Phi^t_{u,v}$ on $\mathbb{T}^2$

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Thus, to study the billiards dynamics on a rectangular table, it is sufficient to understand linear flows on the standard torus.

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2. otherwise, every orbit of the flow is *dense* and even *equidistributed* in $\mathbb{T}^2$: this means that, for any continuous function φ on $\mathbb{T}^2$ and any initial condition $(x_0, y_0) \in \mathbb{T}^2$, we have

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(\Phi^t_{u,v}(x_0, y_0))\, dt = \int_{\mathbb{T}^2} \varphi(x, y)\, dx\, dy.$$

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- when $\frac{u}{v} = \frac{p}{q}$ with integers p, q satisfying $p \wedge q = 1$, we write $u = wp, v = wq$. The period is $\frac{1}{|w|}$.

Sketch of proof in the irrational case

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with $\varphi_{0,0} = \int_{\mathbb{T}^2} \varphi(x,y)\,dx\,dy$ and

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It follows that

$$\begin{aligned}\int_0^T \varphi(\Phi^t_{u,v}(x_0,y_0))\,dt &= T\varphi_{0,0} + \int_0^T \frac{d}{dt}\psi(\Phi^t_{u,v}(x_0,y_0))\,dt \\ &= T\varphi_{0,0} + \psi(\Phi^T_{u,v}(x_0,y_0)) - \psi(x_0,y_0).\end{aligned}$$

Thus, we have the estimate

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For a general continuous function φ on $\mathbb{T}^2$, one uses the case of trigonometric polynomials and (a particular case of) **Stone-Weierstrass theorem**: any continuous function can be **uniformly approximated** by a trigonometric polynomial (details on blackboard if available; exercise otherwise).

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which allows to define the coefficients $\psi_{j,k}$ as above, but the formal Fourier series $\sum_{(j,k)\neq(0,0)} \psi_{j,k} \exp 2\pi i(jx+ky)$ does not always correspond to a true function ψ!

Diophantine numbers

Definition: An irrational number α is *diophantine* if there exists $\tau \geq 0$, $\gamma > 0$, such that, for all $(j, k) \neq (0, 0)$ in $\mathbb{Z}^2$, one has

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Any irrational real algebraic number is diophantine: actually, it satisfies the above condition for any $\tau > 0$ (and appropriate $\gamma = \gamma(\tau)$) ; this is the content of Roth's theorem..

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One has then

$$|\int_0^T \varphi(\Phi^t_{u,v}(x_0, y_0))\, dt| \leqslant 2 \max_{\mathbb{T}^2} |\psi|.$$

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- In the diophantine irrational case, one has very good estimates for the Birkhoff averages of smooth functions.

Thanks for your attention