The arctic circle

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What is a domino?

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A domino is a rectangle of dimensions 2×1, oriented either vertically or horizontally.

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A domino tiling of a domain is a way to cover it with dominos without overlaps.

What is a random domino tiling?

For a given domain, there are only finitely many ways (possibly none at all) to cover is with dominos. Assuming that there are coverings, we can pick one at random, giving the same probability to all of them. This is what we will always mean by a random domino tiling.

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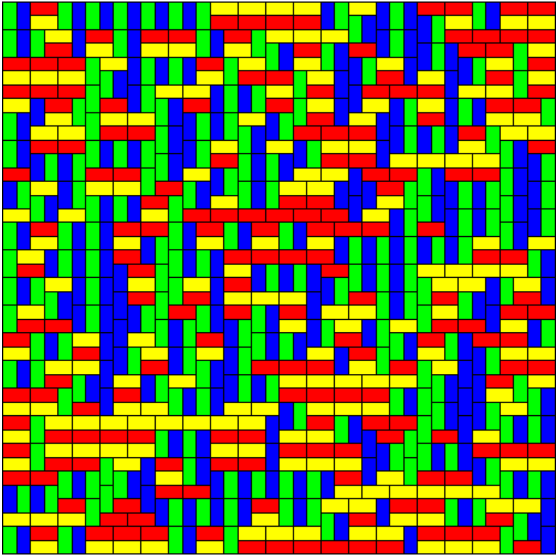

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5. Except, stop after a long enough time.

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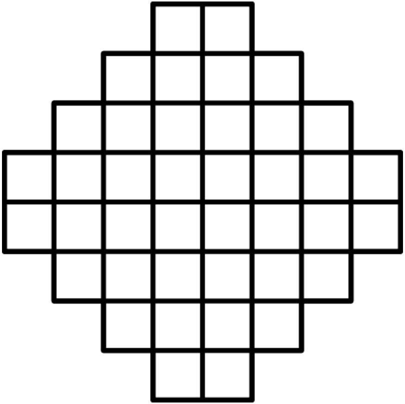

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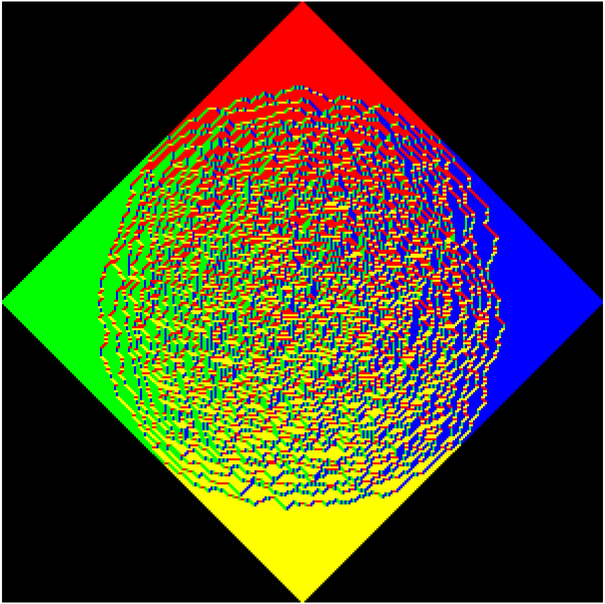

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The "right answer" to the last question holds the key to all the others.

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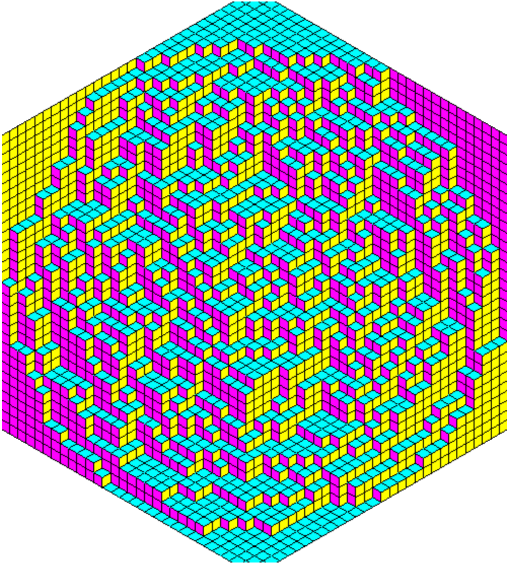

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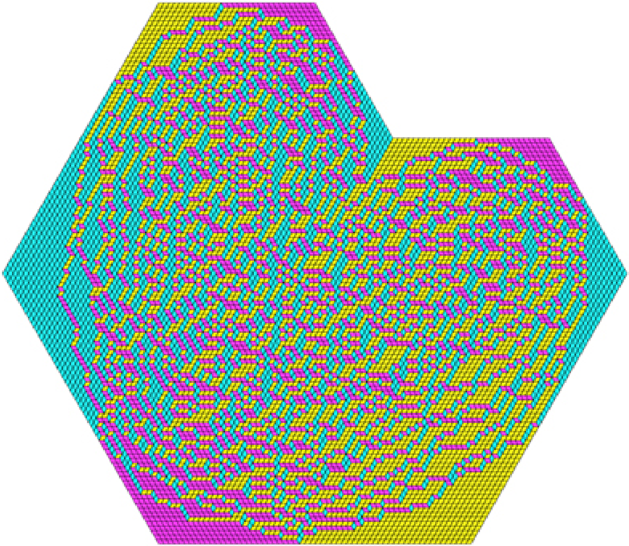

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- What kind of mathematical tools can we use?

The main idea is that, since we select one out of many tilings, the main step will involve counting tilings having a certain behavior.

Easy test: Parity of the number of squares

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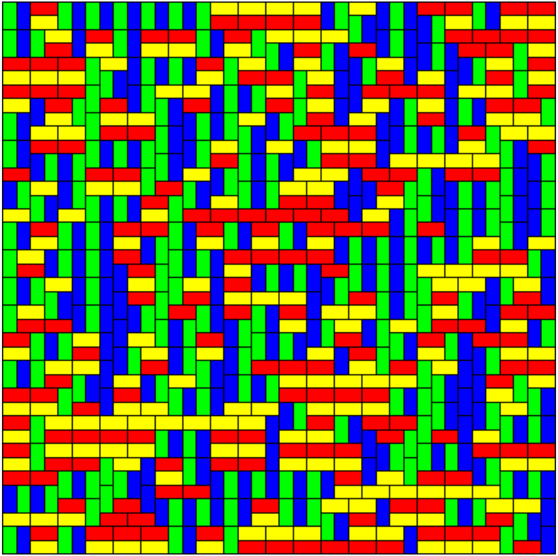
If a domain has a different number of white and black squares, it cannot be covered by dominos.

We will keep this trick from now on: the domain is assumed to be drawn on a chess board. This explains the difference in colors between the dominos in the previous picture:

There are really 4 types of dominos



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$$c = \exp\left(\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\right) \simeq 5.354\ldots$$

The height function

A domino tiling is a complicated mathematical object. We would like to encode it into a simpler one...

The height function is defined by a simple local rule: around a black square, turning counterclockwise, it increases by 1 along every edge, except when crossing a domino (in which case it decreases by 3).

Of course one needs to check that this is well-defined (that there is such a function).

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- On the Aztec diamond, it looks like a jigsaw;
- On the fake Aztec diamond, it looks like a wedge.

Because a height function cannot oscillate much, if the boundary condition is very "wild", then it has little choice inside the domain.

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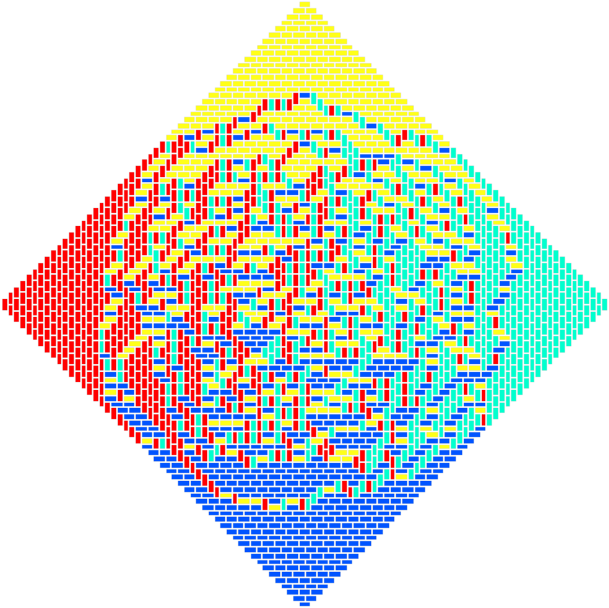

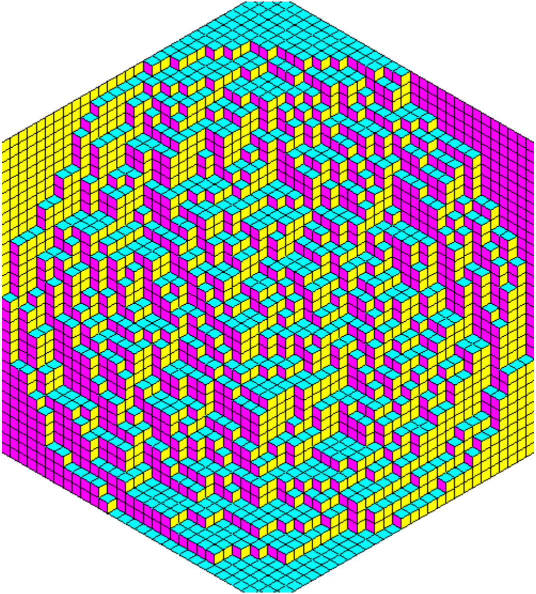

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It is still the case that the number of tilings inside is exponential in the surface of the square, like before. This can be shown using sub-additivity. But the constant is different!

$$N \simeq c(a, b)^{(2n)^2}.$$

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For completeness:

$$c(a, b) = \exp \frac{L(\pi p_1) + L(\pi p_2) + L(\pi p_3) + L(\pi p_4)}{\pi} \quad \text{where}$$

- $L(x) = -\int_0^x \log 2 \sin t dt$ is the Lobachevsky function;
- $2(p_1 - p_2) = a$;
- $2(p_4 - p_3) = b$;
- $p_1 + p_2 + p_3 + p_4 = 1$;
- $\sin(\pi p_1) \sin(\pi p_2) = \sin(\pi p_3) \sin(\pi p_4)$.

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To put things in a formula: The number of tilings giving rise to a height function close to h is approximately

$$N(h) \simeq \exp\left[\iint \log\left(c(\partial_x h, \partial_y h)\right) dxdy\right] \simeq \mathcal{C}(h)^{n^2}.$$

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Theorem

The height function of a uniform random domino tiling, in any domain, is very likely to be close to h_0.

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But: if $c_1 < c_2$, then $c_1^{n^2} \ll c_2^{n^2}$. So the sum below is dominated by one term, maximizing $\mathcal{C}(h')$:

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- If $h = h_0$, then $P[H \simeq h] \to 1$;
- If $h \neq h_0$, then it is exponentially small, like

$$P[H \simeq h] \simeq e^{-c(h)n^2}$$

where $c = -\log(\mathcal{C}(h)/\mathcal{C}(h_0)) > 0$.

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$$P[H \simeq h] \simeq \left(\frac{\mathcal{C}(h)}{\mathcal{C}(h_0)} \right)^{n^2}.$$

There are two cases now, giving the proof of the theorem:

- If $h = h_0$, then $P[H \simeq h] \to 1$;
- If $h \neq h_0$, then it is exponentially small, like

$$P[H \simeq h] \simeq e^{-c(h)n^2}$$

where $c = -\log(\mathcal{C}(h)/\mathcal{C}(h_0)) > 0$.

This is known as a large deviation theorem.